AGMON–KATO–KURODA THEOREMS FOR A LARGE CLASS OF PERTURBATIONS

ALEXANDRU D. IONESCU AND WILHELM SCHLAG

Abstract. We prove asymptotic completeness for operators of the form $H = -\Delta + L$ on $L^2(\mathbb{R}^d)$, $d \geq 2$, where L is an *admissible perturbation*. Our class of admissible perturbations contains multiplication operators defined by real-valued potentials $V \in L^q(\mathbb{R}^d)$, $q \in [d/2, (d+1)/2]$ (if $d = 2$ then we require $q \in (1, 3/2]$), as well as real-valued potentials V satisfying a global Kato condition. The class of admissible perturbations also contains first order differential operators of the form $\vec{a} \cdot \nabla - \nabla \cdot \overline{\vec{a}}$ for suitable vector potentials a. Our main technical statement is a new limiting absorption principle which we prove using techniques from harmonic analysis related to the Stein-Tomas restriction theorem.

1. Introduction

One of the basic problems of quantum mechanics is to determine the spectrum and the spectral types of the self-adjoint operator

$$H = -\Delta + L$$

on $L^2(\mathbb{R}^d)$, where L is a suitable perturbation. A minimal requirement for self-adjointness is that L is symmetric. Given the self-adjoint operator H, let $\sigma_{\mathrm{ac}}(H)$, $\sigma_{\mathrm{sc}}(H)$, and $\sigma_{\mathrm{pp}}(H)$, denote its absolutely continuous spectrum, singular continuous spectrum, and pure point spectrum respectively. Let $H = \int \lambda \, E(d\lambda)$ denote the spectral resolution of H. It is well-known that there is a Lebesgue decomposition

$$E = E_{\mathrm{ac}} + E_{\mathrm{sc}} + E_{\mathrm{pp}},$$

where the terms on the right-hand side are projection valued measures. The ranges of $E_{\mathrm{ac}}(\mathbb{R})$, $E_{\mathrm{sc}}(\mathbb{R})$, and $E_{\mathrm{pp}}(\mathbb{R})$ are orthogonal and are typically denoted by $L^2_{\mathrm{ac}}, L^2_{\mathrm{sc}}$, and L^2_{pp}, respectively. These subspaces are referred to as the absolutely continuous, singular continuous, and pure point subspaces, respectively. Physically, it is most relevant to determine which of these is nonzero. This is related to the long-time behavior of the evolution e^{-itH}. Indeed, any

1991 *Mathematics Subject Classification.* 35P05 and 47A10.

Both authors were supported in part by NSF grants and Alfred P. Sloan research fellowships.

$f \in L^2(\mathbb{R}^d)$ with $E_{\mathrm{pp}} f = f$ does not propagate, whereas $(E_{\mathrm{ac}} + E_{\mathrm{sc}})f = f$ leads to transport (see the RAGE theorem in [2]).

A much-studied case of perturbations are those defined by multiplication with suitable potentials V. For example, if for some $\varepsilon > 0$

$$\sup_{x \in \mathbb{R}^d} (1 + |x|)^{1+\varepsilon} |V(x)| < \infty, \tag{1.1}$$

then a classical theorem of S. Agmon [1] (which applies to all dimensions $d \geq 1$), combined with T. Kato's theorem [8] on absence of eigenvalues in $(0, \infty)$ for such V, states there is *asymptotic completeness* in this case. In dynamical terms, this refers to the fact that for any $f \in L^2(\mathbb{R}^d)$ there is $f_0 \in L^2(\mathbb{R}^d)$ so that

$$e^{-itH} f = \sum_j e^{-it\lambda_j} P_j f + e^{-itH_0} f_0 + o_{L^2}(1)$$

as $t \to \infty$. Here $H_0 = -\Delta$, $\lambda_j \leq 0$ are the eigenvalues of H, and P_j are the orthogonal projections onto the associated eigenspaces. In spectral terms, this means that $E_{\mathrm{sc}} = 0$ and that the wave operators

$$\Omega^{\pm} := \text{s-lim}_{t \to \mp\infty} e^{iHt} e^{-itH_0}$$

exist and are complete, i.e., they are surjective onto the absolutely continuous spectral subspace L^2_{ac} of H. S. Agmon's work was the culmination of a series of partial results for which we refer to [1] and M. Reed, B. Simon [15]. In particular, Agmon deduced the existence and completeness of the wave operators from the *limiting absorption principle*

$$\sup_{\lambda \geq \lambda_0} \sup_{\epsilon > 0} \|(H - (\lambda + i\epsilon))^{-1}\|_{L^{2,\sigma} \to L^{2,-\sigma}} \leq C(V, \lambda_0), \tag{1.2}$$

$\lambda_0 > 0$ and $\sigma > 1/2$, via Kato's smoothing theory, see [9]. Here

$$L^{2,\sigma}(\mathbb{R}^d) := \big\{ f \,:\, (1 + |x|)^\sigma f(x) \in L^2(\mathbb{R}^d) \big\}.$$

We remark that (1.2) immediately leads to the fact that

$$E_{\mathrm{sc}}(\mathbb{R}^+) = 0$$

because of the density of $L^{2,\sigma}$ in L^2, see Theorem XIII.20 in [15]. A recent example of A. Kiselev [12] (in $d = 1$) shows that S. Agmon's theorem is essentially sharp as far as the decay of V is concerned. For a recent review of much of what is known about the spectral theory of decaying potentials we refer to S. Denisov, A. Kiselev's survey [3].

The optimality of (1.1) is related to the optimality of $\sigma > 1/2$ in the limiting absorption principle (1.2). When $V \equiv 0$, the limiting absorption principle (1.2) is intimately connected to basic *restriction theorems* for the Fourier transform. The relevant restriction theorem in this case is the bound

$$\|\hat{f}\|_{L^2(\mathbb{S}^{d-1})} \leq C \|f\|_{L^{2,\sigma}(\mathbb{R}^d)}$$

with $\sigma > 1/2$, known as the *trace-lemma*.

The trace lemma applies to the restriction of the Fourier transform to any compact hypersurface. In particular, it does not use the fact that the Gaussian curvature of the sphere does not vanish. In contrast, the well-known Stein-Tomas restriction theorem asserts that

$$\|\hat{f}\|_{L^2(\mathbb{S}^{d-1})} \leq C\|f\|_{L^{p_d}(\mathbb{R}^d)}$$

where $p_d = (2d+2)/(d+3)$ and $d \geq 2$ (see [19]). This is an optimal bound in the sense that it fails for any $p > p_d$. Moreover, it fails for surfaces with one vanishing principal curvature. It is natural to ask what kind of Agmon-type theorem or limiting absorption principle results from using the Stein-Tomas theorem rather than the much simpler trace lemma. This issue was addressed by M. Goldberg and the second author [5] who obtained the bound

$$\sup_{0<\epsilon<1,\ \lambda\geq\lambda_0} \left\|(-\Delta + V - (\lambda^2 + i\epsilon))^{-1}\right\|_{L^{4/3}\to L^4} \leq C(\lambda_0, V)\ \lambda^{-1/2} \tag{1.3}$$

provided $V \in L^{3/2}(\mathbb{R}^3) \cap L^{3/2+\delta}(\mathbb{R}^3)$, $\delta > 0$. In particular, the spectrum of $-\Delta + V$ is purely absolutely continuous on $(0, \infty)$ for such V. This result depended on the recent unique continuation theorem of the first author and D. Jerison [7], who established the absence of imbedded point spectrum for H under the condition $V \in L^{3/2}(\mathbb{R}^3)$ (with suitable analogues in all dimensions $d \geq 2$). Because of its dependence on a strong unique continuation result at infinity, the approach of [5] was rather limited. In particular, it applied only to potentials $V \in L^{3/2}(\mathbb{R}^3)\cap L^{3/2+\delta}(\mathbb{R}^3)$, $\delta > 0$. Moreover, in [5] no unconditional statement could be made about absence of singular continuous spectrum for $V \in L^p(\mathbb{R}^3)$, $3/2 \leq p \leq 2$.

The goal of this paper is to prove an Agmon-type theorem for a much larger class of perturbations without relying on any unique continuation theorem at infinity. Similar to [1], we will prove a suitable limiting absorption bound. However, extensive use is made of bounds on oscillatory integrals in the spirit of the Stein-Tomas restriction theorem and related bounds for Bochner-Riesz means, see [18, Chapter IX]. We now describe our results in more detail. Our main theorem is Theorem 1.3.

We assume from now on that the dimension d is ≥ 2. We define the sets $D_j = \{x \in \mathbb{R}^d : |x| \in [2^{j-1}, 2^j]\}$, $j \geq 1$, and $D_0 = \{x \in \mathbb{R}^d : |x| \in [0, 1]\}$. Following the notation in [6, Chapter XIV], we also define the following Banach spaces of functions on $\mathbb{R}^d$, $d \geq 2$:

$$B = \Big\{f : \mathbb{R}^d \to \mathbb{C} : ||f||_B := \sum_{j=0}^{\infty} 2^{j/2}||f||_{L^2(D_j)} < \infty\Big\},$$

$$B^* = \Big\{u : \mathbb{R}^d \to \mathbb{C} : ||u||_{B^*} := \sup_{j\geq 0} 2^{-j/2}||u||_{L^2(D_j)} < \infty\Big\}.$$

The spaces B and B^* are related to the sharp form of the trace lemma

$$\mathcal{F} : B \to L^2(\mathbb{S}^{d-1}) \text{ and } \mathcal{F}^{-1} : L^2(\mathbb{S}^{d-1}) \to B^*, \tag{1.4}$$

as bounded operators (see [6, Theorem 7.1.26]).

Let $\mathcal{S}(\mathbb{R}^d)$ denote the space of Schwartz functions on $\mathbb{R}^d$ and $\mathcal{S}'(\mathbb{R}^d)$ the space of distributions. For any $\alpha \in \mathbb{C}$ let $S_\alpha : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ denote the operator defined by the Fourier multiplier $\xi \to (1 + |\xi|^2)^{\alpha/2}$, i.e., $S_\alpha = (1 - \Delta)^{\alpha/2}$. For $1 < p < \infty$ and $\alpha \in \mathbb{R}$ we define the standard Sobolev spaces

$$W^{\alpha,p} = \{u \in \mathcal{S}'(\mathbb{R}^d) : S_\alpha u \in L^p\} \text{ with } ||u||_{W^{\alpha,p}} := ||S_\alpha u||_{L^p}.$$

Let $p_d = (2d + 2)/(d + 3)$ and $p'_d = (2d + 2)/(d - 1)$ denote the Stein-Tomas restriction exponents. Let $S_1(B)$ denote the image of B under S_1, and $S_{-1}(B^*)$ the image of B^* under S_{-1}. The main Banach spaces we use in this paper are

$$X := W^{-1/(d+1),p_d} + S_1(B) \text{ with } ||f||_X := \inf_{f_1+f_2=f} ||S_{-1/(d+1)} f_1||_{L^{p_d}} + ||S_{-1} f_2||_B,$$

and

$$X^* := W^{1/(d+1),p'_d} \cap S_{-1}(B^*) \text{ with } ||u||_{X^*} := \max(||S_{1/(d+1)} u||_{L^{p'_d}}, ||S_1 u||_{B^*}).$$

Clearly, X is a space of distributions and $X^* \subseteq W^{1,2}_{\mathrm{loc}}$. To motivate these definitions, we notice first that

$$\mathcal{F} : X \to L^2(\mathbb{S}^{d-1}) \text{ and } \mathcal{F}^{-1} : L^2(\mathbb{S}^{d-1}) \to X^*, \tag{1.5}$$

as bounded operators, which is a consequence of the Stein-Tomas restriction theorem and (1.4). Moreover

$$X \hookrightarrow W^{-1,2} \text{ and } W^{1,2} \hookrightarrow X^*, \tag{1.6}$$

which follows from the Sobolev imbedding theorem (this explains the choice of the exponent $1/(d + 1)$ in the definition of X and X^*) and the imbedding $B \hookrightarrow L^2 \hookrightarrow B^*$. Finally, for more general theorems, we would like to have the space X as large as possible and the space X^* as small as possible, subject to (1.5) and (1.6). Our first theorem is a uniform bound for the free resolvent $R_0(z)$, $z \in \mathbb{C} \setminus [0, \infty)$. The operator $R_0(z)$ is defined on $\mathcal{S}'(\mathbb{R}^d)$ by the Fourier multiplier $\xi \to (|\xi|^2 - z)^{-1}$.

Theorem 1.1. *Assume that $\delta \in (0, 1]$. Then*

$$\sup_{|\lambda|\in[\delta,\delta^{-1}],\epsilon\in[-1,1]\setminus\{0\}} ||R_0(\lambda + i\epsilon)||_{X\to X^*} \le C_\delta < \infty, \tag{1.7}$$

where C_δ is a (finite) constant that depends only on δ and the dimension d.

The main point of Theorem 1.1 is the uniformity of the bound (1.7) as $\epsilon \to 0$. In contrast, the bound in the stronger (elliptic) imbedding $R_0(\lambda+i\epsilon) : W^{-1,2} \to W^{1,2}$ blows up as $\epsilon \to 0$ if $\lambda > 0$. The proof of Theorem 1.1 is essentially known

through the work of L. Hörmander, C. Kenig, A. Ruiz, C. Sogge, and L. Vega, see [6], [10], [16]; we collect the necessary bounds in Section 2.

We also prove a weighted estimate. For $N \geq 0$, $\gamma \in (0,1]$, and $x \in \mathbb{R}^d$, we define the weight

$$\mu_{N,\gamma}(x) = \frac{(1+|x|^2)^N}{(1+\gamma|x|^2)^N}. \tag{1.8}$$

Theorem 1.2. *Assume that $\delta \in (0,1]$. Then*

$$||\mu_{N,\gamma}u||_{X^*} \leq C_{N,\delta}||\mu_{N,\gamma}(\Delta+\lambda)u||_X \tag{1.9}$$

for any $\lambda \in \mathbb{R}$ with $|\lambda| \in [\delta, \delta^{-1}]$, and any $u \in X^*$ with the property that

$$\lim_{R\to\infty} R^{-1} \int_{R\leq|x|\leq 2R} |u|^2\, dx = 0. \tag{1.10}$$

The constant $C_{N,\delta}$ depends only on N, δ, and the dimension d.

We remark that the condition (1.10) is necessary: let

$$u(x) = \int_{\mathbb{S}^{d-1}} e^{-ix\cdot\xi}\, d\xi.$$

Then $u \in X^*$, however $(\Delta+1)u \equiv 0$. Theorem 1.2 plays a key role in the bootstrap argument in the proof of our main Theorem 1.3 below (see Lemma 4.4). We emphasize that the constant in (1.9) is allowed to depend on the parameter N, but not on $\gamma \in (0,1]$.

For functions $u, f \in \mathcal{S}(\mathbb{R}^d)$, we define $\langle u, f\rangle := \int_{\mathbb{R}^d} u\overline{f}\, dx$. Clearly, $\langle u, f\rangle = \langle S_\alpha(u), S_{-\alpha}(f)\rangle$ for any $\alpha \in \mathbb{R}$. By a slight abuse of notation, we extend the definition of $\langle ., .\rangle$ to pairs

$$(u,f) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \cup L^2 \times L^2 \cup W^{-1,2} \times W^{1,2} \cup X \times X^*.$$

We have

$$|\langle u, f\rangle| \leq \min(||u||_{L^2}||f||_{L^2}, ||u||_{W^{-1,2}}||f||_{W^{1,2}}, ||u||_X||f||_{X^*}). \tag{1.11}$$

Also, it follows easily from the definitions of the spaces X and X^* that

$$||f||_X \leq C \sup_{\phi\in\mathcal{S}(\mathbb{R}^d),\, ||\phi||_{X^*}=1} |\langle f, \phi\rangle| \text{ and } ||u||_{X^*} \leq C \sup_{\phi\in\mathcal{S}(\mathbb{R}^d),\, ||\phi||_X=1} |\langle u, \phi\rangle|. \tag{1.12}$$

Definition: *Let $\mathcal{L}(X^*, X)$ denote the space of bounded operators from X^* to X. We say that L is an* admissible perturbation *if:*

(1) $L \in \mathcal{L}(X^*, X)$ and

$$\langle L\phi, \psi\rangle = \overline{\langle L\psi, \phi\rangle} \tag{1.13}$$

for any $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$ (i.e. L is symmetric).

(2) For any $\varepsilon > 0$ and $N \geq 0$ there are $A_{N,\varepsilon}, R_{N,\varepsilon} \in [1,\infty)$ such that

$$||\mu_{N,\gamma}Lu||_X \leq \varepsilon||\mu_{N,\gamma}u||_{X^*} + A_{N,\varepsilon}||u\mathbf{1}_{\{|x|\leq R_{N,\varepsilon}\}}||_{L^2} \tag{1.14}$$

for any $u \in X^*$ and any $\gamma \in (0,1]$, where $\mathbf{1}_E$ denotes the characteristic function of the set E.

(3) There is an integer $J \geq 1$ and operators A_j, B_j in $\mathcal{L}(X^*, L^2)$ for $1 \leq j \leq J$ such that

$$\langle L\phi, \psi\rangle = \sum_{j=1}^{J} \langle B_j\phi, A_j\psi\rangle \text{ for any } \phi, \psi \in X^*. \tag{1.15}$$

Moreover, considered as (unbounded) operators on L^2, A_j, B_j are closed on some domains satisfying[1]

$$\mathrm{Domain}(A_j) \supseteq W^{1,2}(\mathbb{R}^d), \quad \mathrm{Domain}(B_j) \supseteq W^{1,2}(\mathbb{R}^d)$$

for all $1 \leq j \leq J$.

Our main goal is to prove an Agmon-type theorem for admissible perturbations. Before formulating our main theorem, we remark that condition (1) in the definition of admissible perturbations is essential for our arguments. Condition (2) is somewhat technical and is related to our use of Theorem 1.2. Variations (and improvements) of condition (2) are possible. Condition (3) is the usual condition which arises in Kato's smoothing theory [9], and is needed in order to study the wave operators which intertwine $-\Delta$ and $-\Delta + L$.

In view of (1.6),

$$H := -\Delta + L : W^{1,2} \to W^{-1,2}$$

as a bounded operator, for any admissible perturbation L. Our main theorem is the following:

Theorem 1.3. *Assume that L is an admissible perturbation. Then the following properties hold:*

(a) The operator $H = -\Delta + L$ defines a closed, self-adjoint operator on

$$\mathrm{Domain}(H) = \{u \in W^{1,2}(\mathbb{R}^d) : Hu \in L^2(\mathbb{R}^d)\}. \tag{1.16}$$

In addition, $\mathrm{Domain}(H)$ *is dense in $L^2(\mathbb{R}^d)$ and H is bounded from below on* $\mathrm{Domain}(H)$.

(b) The set of nonzero eigenvalues $\mathcal{E} = \sigma_{\mathrm{pp}} \setminus \{0\}$ of H is discrete in $\mathbb{R} \setminus \{0\}$, i.e., $\mathcal{E} \cap I$ is finite for any compact set $I \subset \mathbb{R} \setminus \{0\}$. Moreover, each eigenvalue in $\mathcal{E}$ has finite multiplicity.

(c) Any eigenfunction u of H with eigenvalue $\lambda \neq 0$ is rapidly decreasing, i.e., for any integer $N \geq 0$,

$$(1 + |x|^2)^N u \in W^{1,2}(\mathbb{R}^d). \tag{1.17}$$

(d) Let $I \subset (\mathbb{R} \setminus \{0\}) \setminus \mathcal{E}$ be compact. Then

$$\sup_{\lambda \in I, \epsilon \in [-1,1]\setminus 0} ||R_L(\lambda + i\epsilon)||_{X \to X^*} \leq C(L, I) < \infty, \tag{1.18}$$

[1] These inclusions are natural since $W^{1,2} \subset X^*$

where $R_L(\lambda+i\epsilon)$ denotes the resolvent of H at $\lambda+i\varepsilon$, and $C(L,I)$ is a constant that depends on the interval I, the perturbation L, and the dimension d. Thus the spectrum of the operator H is purely absolutely continuous on I.

(e) $\sigma_{\mathrm{sc}}(H)=\emptyset$ and $\sigma_{\mathrm{ac}}(H)=[0,\infty)$.

(f) The wave operators $\Omega^{\pm}(H,H_0)$ exist and are complete, where $H_0=-\Delta$.

We notice the similarity of Theorem 1.3 with the Agmon-Kato-Kuroda theorem, see Theorem XIII.33 in [15]. The main novelty in our theorem is that it applies to a much larger class of perturbations. To provide examples of admissible perturbations we define the Banach space

$$Y=\Big\{V:\mathbb{R}^d\to\mathbb{C}:||V||_Y:=\sum_{j=0}^{\infty}2^j||f||_{L^\infty(D_j)}<\infty\Big\},$$

where the sets D_j are as in the definitions of the spaces B and B^*. For $\delta\in(0,1/2]$ we define the kernels

$$K_{d,\delta}(x)=\mathbf{1}_{\{|x|\le\delta\}}\begin{cases}|x|^{-(d-2)} & \text{if } d\ge 3;\\ \log(1/|x|) & \text{if } d=2.\end{cases}$$

For any exponent $q\in[1,\infty)$ and measurable function f let

$$M_q(f)(x)=\Big[\int_{|y|\le 1/2}|f(x+y)|^q\,dy\Big]^{1/q}.$$

Clearly, $M_q(f)(x)\le CM_{q'}(f)(x)$ if $1\le q\le q'\le\infty$. Also, $||M_q(f)||_{L^{p'}(D_j)}\le C||M_q(f)||_{L^p(\widetilde{D}_j)}$ if $1\le p\le p'\le\infty$, where $\widetilde{D}_j=D_{j-1}\cup D_j\cup D_{j+1}$ if $j\ge 1$ and $\widetilde{D}_0=D_0\cup D_1$ (the last inequality is easy to prove for $p'=p$ and $p'=\infty$, thus for $p'\in[p,\infty]$ by interpolation). We fix $q_0=d/2$ if $d\ge 3$ and $q_0>1$ if $d=2$.

Proposition 1.4. *The following are examples of admissible perturbations:*

(a) Multiplication operators defined by real-valued potentials V with the property that

$$M_{q_0}(V)\in L^{(d+1)/2}, \tag{1.19}$$

or

$$M_{q_0}(V)\in Y, \tag{1.20}$$

or

$$\lim_{\delta\to 0}||\,|V|*K_{d,\delta}||_Y=0. \tag{1.21}$$

(b) First order differential operators of the form

$$\vec{a}\cdot\nabla-\nabla\cdot\overline{\vec{a}},$$

defined by vector-valued potentials $\vec{a}:\mathbb{R}^d\to\mathbb{C}^d$ with the property that

$$\Big[\sum_{j=0}^{\infty}\Big(2^{j/2}||M_{2q_0}(|\vec{a}|)||_{L^{d+1}(D_j)}\Big)^{p(d)}\Big]^{1/p(d)}<\infty, \tag{1.22}$$

or

$$M_{2q_0}(|\vec{a}|)\in Y, \tag{1.23}$$

or

$$\lim_{\delta\to 0}||\,[|\vec{a}|^2 * K_{d,\delta}]^{1/2}||_Y=0. \tag{1.24}$$

(c) Any finite linear combination of admissible perturbations with real coefficients.

We remark that the exponent $(d+1)/2$ in (1.19) is optimal for Theorem 1.3 to hold. This is due to a recent example by the first author and D. Jerison [7] of a potential $V\in L^p$, for all $p>(d+1)/2$, such that $H=-\Delta+V$ has slowly decaying eigenfunctions (and positive eigenvalues). We emphasize that this example is not related to the local singularities of V. In fact, V is a smooth, real-valued function with oscillations and asymptotic behavior

$$|V(x)|\approx(1+|x_1|+|x'|^2)^{-1}.$$

The main issue here is that the potential behaves differently along different directions. It remains to be seen if such examples can lead to dense point spectrum or even imbedded singular continuous spectrum as well. It is possible that the transition point for singular continuous spectrum occurs at larger values of q, for example at $q=d$ (Coulomb case)[2]. The same remark applies to first order perturbations defined by vector potentials a as in (1.22). The restriction on the exponent q_0 is needed to define the operator H as a self-adjoint operator on its domain.

In some cases of admissible perturbations we can add the natural conclusion

$$\sigma_{pp}\subseteq(-\infty,0]. \tag{1.25}$$

For potentials $V\in L^{d/2}(\mathbb{R}^d)$, $d\geq 3$ (a restriction stronger than (1.19)), this follows from [7, Theorem 2.1]. H. Koch and D. Tataru [13] have recently proved the absence of positive eigenvalues for potentials V that satisfy conditions similar to (1.19).

We also allow perturbations given by multiplication with potentials in the global Kato class described in (1.21). For comparison, the *local Kato class* (cf. [17]) is defined by the condition

$$\lim_{\delta\to 0}||\,|V|*K_{d,\delta}||_{L^\infty}=0.$$

[2]This possibility was communicated to the second author by Barry Simon, who believes that it should be $q=d$.

We remark that the condition (1.21) is more general than S. Agmon's condition (1.3) in [1]

$$\sup_{x\in\mathbb{R}^d}\left[(1+|x|)^{2+2\epsilon}\int_{|y-x|\leq 1}|V(y)|^2|y-x|^{-d+\mu}\,dy\right]<\infty.$$

for some $\epsilon>0$ and $0<\mu<4$. This is easy to see, using the Cauchy–Schwartz inequality and the fact that $(1+|x|)^{-1-\epsilon}\in Y$. For operators defined by potentials V as in (1.21), the conclusion (1.25) is not known; even the easier question of absence of compactly supported eigenfunctions for such potentials is not settled (see [17, p. 519]). Our proof of Theorem 1.3 is independent of the validity of (1.25).

We have the following corollary of Theorem 1.3, which relies on the well-known connection between smoothing bounds for the resolvent as in (1.18) and time-dependent smoothing bounds. This connection is given by T. Kato's theory [9].

Corollary 1.5. *Let L be an admissible perturbation, and let $\mathcal{E}$ be as in Theorem 1.3. Then for any compact set $I\subseteq(\mathbb{R}\setminus\{0\})\setminus\mathcal{E}$, there exists a constant $C(I,L)$ such that*

$$||S_{1/(d+1)}[e^{itH}E(I)f]||_{L_x^{p_d'}L_t^2}+||S_1[e^{itH}E(I)f]||_{B_x^*L_t^2}\leq C(I,L)\|f\|_2, \tag{1.26}$$

for any $f\in L^2$, where $E(I)$ denotes the spectral projection onto the interval I associated with H.

Because of the spectral projection $E(I)$ with a compact I the smoothing effect in Corollary 1.5 given by derivatives is less meaningful. Nevertheless, we state it in this form since it is directly related to (1.18).

The rest of the paper is organized as follows: in Section 2 we prove Theorem 1.1. In Section 3 we prove Theorem 1.2, which is essential for our bootstrap argument in the proof of the main Theorem 1.3 and also leads to the rapid decay of eigenfunctions with nonzero eigenvalues in (1.17). In Section 4 we transfer the limiting absorption principle (1.18) from the case $L=0$ (Theorem 1.1) to the general case of admissible perturbations, by means of the resolvent identity and Fredholm's alternative. In Section 5 we prove Theorem 1.3 and Corollary 1.5. Finally, in Section 6 we prove Proposition 1.4.

The authors would like to thank C. E. Kenig and B. Simon for useful discussions.

2. Proof of Theorem 1.1

For $\lambda\in[-\delta^{-1},-\delta]$ or $|\epsilon|>\delta$, we have the elliptic bound

$$||R_0(\lambda+i\epsilon)||_{W^{-1,2}\to W^{1,2}}\leq C_\delta,$$

which is stronger than (1.7), in view of (1.6). Thus, we may assume that $\lambda \in [\delta, \delta^{-1}]$ and $|\epsilon| \le \delta$. Let $\chi : \mathbb{R}^d \to [0,1]$ denote a smooth function supported in the set $\{|\xi| \in [\sqrt{\lambda}/2, 3\sqrt{\lambda}/2]\}$ and equal to 1 in the set $\{|\xi| \in [3\sqrt{\lambda}/4, 5\sqrt{\lambda}/4]\}$. Let $\chi(D)$ and $(1-\chi)(D)$ denote the operators defined by the Fourier multipliers $\xi \to \chi(\xi)$ and $\xi \to 1-\chi(\xi)$. For the operator $(1-\chi)(D)R_0(\lambda + i\epsilon)$ we have again the stronger elliptic bound

$$||(1-\chi)(D)R_0(\lambda + i\epsilon)||_{W^{-1,2}\to W^{1,2}} \le C_\delta.$$

It remains to prove that

$$||\chi(D)R_0(\lambda + i\epsilon)||_{X\to X^*} \le C_\delta,$$

which is equivalent to

$$\begin{aligned} &||S_1\chi(D)R_0(\lambda+i\epsilon)S_1||_{B\to B^*} + ||S_{\frac{1}{d+1}}\chi(D)R_0(\lambda+i\epsilon)S_{\frac{1}{d+1}}||_{L^{p_d}\to L^{p_d'}} \\ &+ ||S_1\chi(D)R_0(\lambda+i\epsilon)S_{\frac{1}{d+1}}||_{L^{p_d}\to B^*} + ||S_{\frac{1}{d+1}}\chi(D)R_0(\lambda+i\epsilon)S_1||_{B\to L^{p_d'}} \le C_\delta. \end{aligned} \tag{2.1}$$

The $B \to B^*$ bound in (2.1) follows from [6, Theorem 14.2.2]. Also, since

$$\langle S_{1/(d+1)}\chi(D)R_0(\lambda+i\epsilon)S_1 f, g\rangle = \langle f, S_1\chi(D)R_0(\lambda - i\epsilon)S_{1/(d+1)}g\rangle$$

for any $f, g \in \mathcal{S}(\mathbb{R}^d)$, it follows that

$$||S_{1/(d+1)}\chi(D)R_0(\lambda+i\epsilon)S_1||_{B\to L^{p_d'}} = ||S_1\chi(D)R_0(\lambda-i\epsilon)S_{1/(d+1)}||_{L^{p_d}\to B^*}.$$

Thus it remains to prove the $L^{p_d} \to L^{p_d'}$ and the $L^{p_d} \to B^*$ bounds in (2.1). The $L^{p_d} \to B^*$ bound follows from the work of A. Ruiz and L. Vega, see [16, Theorem 3.1]. For the $L^{p_d} \to L^{p_d'}$ bound, we notice that $S_{2/(d+1)}\chi(D)$ is a bounded operator on L^{p_d}, so it suffices to prove that

$$||R_0(\lambda+i\epsilon)||_{L^{p_d}\to L^{p_d'}} \le C_\delta$$

uniformly in ϵ and $\lambda \in [\delta, \delta^{-1}]$. This follows from [10, Theorem 2.3].

3. Proof of Theorem 1.2

The constants C_N in this section may depend on N and the dimension d, but not on γ. We start with a lemma concerning the weight $\mu_{N,\gamma}$.

Lemma 3.1. *(a) We have*

$$\partial_{x_j}\mu_{N,\gamma} = \mu_{N,\gamma}b_j, \quad \Delta\mu_{N,\gamma} = \mu_{N,\gamma}b;$$
$$\partial_{x_j}\mu_{N,\gamma}^{-1} = -\mu_{N,\gamma}^{-1}b_j, \quad \Delta\mu_{N,\gamma}^{-1} = \mu_{N,\gamma}^{-1}\widetilde{b},$$

for some functions b_j, b, and $\widetilde{b}$ with the property that for any $x \in \mathbb{R}^n$

$$\sum_{j=1}^d |b_j(x)|(1+|x|^2)^{1/2} + |b(x)|(1+|x|^2) + |\widetilde{b}(x)|(1+|x|^2) \le C_N. \tag{3.1}$$

(b) We have

$$\frac{\mu_{N,\gamma}(x)}{\mu_{N,\gamma}(y)} + \frac{\mu_{N,\gamma}(y)}{\mu_{N,\gamma}(x)} \leq C_N(1+|x-y|^2)^N,$$

for any $x, y \in \mathbb{R}^d$.

(c) For any $r \in (0,\infty)$ *and* $x \in \mathbb{R}^d$,

$$\mu_{N,\gamma}(x) \leq C_{N,r}\mu_{N,\gamma}(rx) \leq C_{N,r}\mu_{N,\gamma}(x).$$

Proof of Lemma 3.1. The proof follows easily from the formula (1.8). □

We also need a technical lemma that allows us to commute the operators S_α and multiplication by the weight $\mu_{N,\gamma}$.

Lemma 3.2. *For* $\alpha \in [-2,2]$

$$||\mu S_\alpha \mu^{-1} S_{-\alpha}||_{L^p\to L^p} + ||\mu S_\alpha \mu^{-1} S_{-\alpha}||_{B\to B} + ||\mu S_\alpha \mu^{-1} S_{-\alpha}||_{B^*\to B^*} \leq C_N, \quad (3.2)$$

and

$$||S_\alpha \mu S_{-\alpha}\mu^{-1}||_{L^p\to L^p} + ||S_\alpha \mu S_{-\alpha}\mu^{-1}||_{B\to B} + ||S_\alpha \mu S_{-\alpha}\mu^{-1}||_{B^*\to B^*} \leq C_N, \quad (3.3)$$

where $p \in \{p_d, 2, p_d'\}$ *and* $\mu \in \{\mu_{N,\gamma}, \mu_{N,\gamma}^{-1}\}$.

Proof of Lemma 3.2. (a) By analytic interpolation, it suffices to prove (3.2) and (3.3) for $\alpha = \pm 2 + i\beta$, $\beta \in \mathbb{R}$, with constant $C_N e^{\beta^2}$. Notice also that

$$\langle \mu S_{-2-i\beta}\mu^{-1}S_{2+i\beta}f, \psi\rangle = \langle f, S_{2-i\beta}\mu^{-1}S_{-2+i\beta}\mu\psi\rangle,$$

for any $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$. So it suffices to prove (3.2) and (3.3) for $\alpha = 2 + i\beta$, $\beta \in \mathbb{R}$. We use the fact that $S_2 = -\Delta + 1$ and Lemma 3.1(a). Then, with $a_j = b_j$ and $a = b$, or $a_j = -b_j$ and $a = \widetilde{b}$

$$\mu S_{2+i\beta}\mu^{-1}S_{-2-i\beta} = \mu S_{i\beta}\mu^{-1}[S_{-i\beta} + 2\sum_j a_j \partial_{x_j} S_{-2-i\beta} + aS_{-2-i\beta}] \quad (3.4)$$

and

$$S_{2+i\beta}\mu S_{-2-i\beta}\mu^{-1} = S_{i\beta}[\mu S_{-i\beta}\mu^{-1} + 2\sum_j a_j \mu \partial_{x_j} S_{-2-i\beta}\mu^{-1} + a\mu S_{-2-i\beta}\mu^{-1}]. \quad (3.5)$$

The operator $S_{i\beta}$ is bounded on B, B^*, and L^p, by the theory of singular integrals (for boundedness on B and B^*, notice that the kernel of this operator is rapidly decreasing at ∞). The same is true for the operator $2\sum_j a_j\partial_{x_j}S_{-2-i\beta} + aS_{-2-i\beta}$ in the right-hand side of (3.4), in view of Lemma 3.1(a). Therefore it suffices to prove that if $m \in C^\infty(\mathbb{R}^d)$ satisfies the differential bounds

$$|\partial_\xi^\nu m(\xi)| \leq C_\nu(1+|\xi|^2)^{-|\nu|/2} \quad (3.6)$$

for any $\xi \in \mathbb{R}^d$ and multi-index ν, then

$$||\mu m(D)\mu^{-1}||_{L^p\to L^p} + ||\mu m(D)\mu^{-1}||_{B\to B} + ||\mu m(D)\mu^{-1}||_{B^*\to B^*} \leq C_{N,m}, \quad (3.7)$$

where p and μ are as in Lemma 3.2, and $m(D)$ denotes the operator defined by the Fourier multiplier $\xi \to m(\xi)$.

To prove (3.7), we use the fact the kernel of the operator $m(D)$ has rapid decay away from the diagonal:

$$|m(D)(x,y)| \leq C_\nu |x-y|^{-\nu} \tag{3.8}$$

for any $x, y \in \mathbb{R}^d$ and integer $\nu \geq 0$. This follows from (3.6) by integration by parts. Let D_j denote the sets in the definition of the spaces B and B^*. We show first that if $p \in \{p_d, 2, p_d'\}$ then

$$||\mathbf{1}_{D_{j'}} \mu m(D) \mu^{-1} \mathbf{1}_{D_j}||_{L^p \to L^p} \leq C_{N,m} 2^{-|j-j'|}, \tag{3.9}$$

for any integers $j, j' \geq 0$. For $|j - j'| \geq 2$ we can simply use (3.8) and the fact that the absolute value of the kernel of the operator $\mathbf{1}_{D_{j'}} \mu m(D) \mu^{-1} \mathbf{1}_{D_j}$ is $\mathbf{1}_{D_{j'}}(x)\mu(x)|m(D)(x,y)|\mu^{-1}(y)\mathbf{1}_{D_j}(y) \leq C_{N,\nu}(1+|x-y|)^{-\nu} 2^{-|j-j'|}$, in view of Lemma 3.1(b). For $|j - j'| \leq 1$ we use the fact that $m(D)$ defines a bounded operator on L^p:

$$\begin{aligned} ||\mathbf{1}_{D_{j'}} \mu m(D) \mu^{-1} \mathbf{1}_{D_j} f||_{L^p} &\leq \sup_{x \in D_{j'}} \mu(x) ||m(D) \mu^{-1} \mathbf{1}_{D_j} f||_{L^p} \\ &\leq \sup_{x \in D_{j'}} \mu(x) \sup_{y \in D_j} \mu^{-1}(y) ||f||_{L^p}, \end{aligned}$$

which proves the $L^p \to L^p$ bound in (3.9), in view of Lemma 3.1(c).

We complete now the proof of (3.7). For the $L^p \to L^p$ bound:

$$\begin{aligned} &||\mu m(D) \mu^{-1} f||_{L^p}^p = \sum_{j'=1}^{\infty} \big|\big| \sum_{j=1}^{\infty} \mathbf{1}_{D_{j'}} \mu m(D) \mu^{-1} \mathbf{1}_{D_j} f \big|\big|_{L^p}^p \\ &\leq C_{N,m}^p \sum_{j'=1}^{\infty} \big[\sum_{j=1}^{\infty} 2^{-|j-j'|} ||\mathbf{1}_{D_j} f||_{L^p} \big]^p \leq C_{N,m}^p \sum_{j=1}^{\infty} ||\mathbf{1}_{D_j} f||_{L^p}^p = C_{N,m}^p ||f||_{L^p}^p, \end{aligned}$$

as desired. The proof of the $B \to B$ and $B^* \to B^*$ bounds is similar, using the $L^2 \to L^2$ bound in (3.9). This completes the proof of Lemma 3.2. $\square$

For later use, we show that if $\chi \in C_0^\infty(\mathbb{R}^d)$ and $\chi(D)$ denotes the operator defined by the Fourier multiplier $\xi \to \chi(\xi)$ then

$$||\mu_{N,\gamma} \chi(D) g||_X \leq C_{N,\chi} ||\mu_{N,\gamma} g||_X \tag{3.10}$$

for any $g \in X$. For this, we notice first that if $g \in X$ then $\mu_{N,\gamma} g \in X$: let $g = g_1 + g_2$ with $S_{-1/(d+1)} g_1 \in L^{p_d}$ and $S_{-1} g_2 \in B$. Then

$$\begin{aligned}
||\mu_{N,\gamma} g||_X &\leq ||S_{-1/(d+1)} \mu_{N,\gamma} g_1||_{L^{p_d}} + ||S_{-1} \mu_{N,\gamma} g_2||_B \\
&= ||[S_{-1/(d+1)} \mu_{N,\gamma} S_{1/(d+1)} \mu_{N,\gamma}^{-1}] \mu_{N,\gamma} S_{-1/(d+1)} g_1||_{L^{p_d}} \\
&+ ||[S_{-1} \mu_{N,\gamma} S_1 \mu_{N,\gamma}^{-1}] \mu_{N,\gamma} S_{-1} g_2||_B \\
&\leq C_{N,\gamma} [||S_{-1/(d+1)} g_1||_{L^{p_d}} + ||S_{-1} g_2||_B] \leq C_{N,\gamma} ||g||_X,
\end{aligned}$$

using Lemma 3.2 and the fact that $|\mu_{N,\gamma}(x)| \leq C_{N,\gamma}$. To prove (3.10), let $g = g_1 + g_2$ be such that $2||\mu_{N,\gamma} g||_X \geq ||S_{-1/(d+1)} \mu_{N,\gamma} g_1||_{L^{p_d}} + ||S_{-1} \mu_{N,\gamma} g_2||_B$. Then

$$\begin{aligned}
&||\mu_{N,\gamma} \chi(D) g||_X \leq ||S_{-1/(d+1)} \mu_{N,\gamma} \chi(D) g_1||_{L^{p_d}} + ||S_{-1} \mu_{N,\gamma} \chi(D) g_2||_B \\
&= ||[S_{\frac{-1}{d+1}} \mu_{N,\gamma} S_{\frac{1}{d+1}} \mu_{N,\gamma}^{-1}][\mu_{N,\gamma} \chi(D) \mu_{N,\gamma}^{-1}][\mu_{N,\gamma} S_{\frac{-1}{d+1}} \mu_{N,\gamma}^{-1} S_{\frac{1}{d+1}}] S_{\frac{-1}{d+1}} \mu_{N,\gamma} g_1||_{L^{p_d}} \\
&+ ||[S_{-1} \mu_{N,\gamma} S_1 \mu_{N,\gamma}^{-1}][\mu_{N,\gamma} \chi(D) \mu_{N,\gamma}^{-1}][\mu_{N,\gamma} S_{-1} \mu_{N,\gamma}^{-1} S_1] S_{-1} \mu_{N,\gamma} g_2||_B \\
&\leq C_{N,\chi} [||S_{-1/(d+1)} \mu_{N,\gamma} g_1||_{L^{p_d}} + ||S_{-1} \mu_{N,\gamma} g_2||_B] \leq C_{N,\chi} ||\mu_{N,\gamma} g||_X,
\end{aligned}$$

using Lemma 3.2 and (3.7). This completes the proof of (3.10).

We turn now to the proof of Theorem 1.2. We prove first the bound (1.9) under the additional restriction

$$u \in \mathcal{S}(\mathbb{R}^d). \tag{3.11}$$

In the case $\lambda \in [-\delta^{-1}, -\delta]$ we prove the stronger elliptic bound

$$||\mu_{N,\gamma} u||_{W^{1,2}} \leq C_{N,\delta} ||\mu_{N,\gamma} (\Delta + \lambda) u||_{W^{-1,2}}. \tag{3.12}$$

Let $f = S_{-1} \mu_{N,\gamma} (\Delta + \lambda) u$. Then $S_1 \mu_{N,\gamma} u = S_1 \mu_{N,\gamma} (\Delta + \lambda)^{-1} \mu_{N,\gamma}^{-1} S_1 f$. Thus

$$\begin{aligned}
&||\mu_{N,\gamma} u||_{W^{1,2}} = ||S_1 \mu_{N,\gamma} u||_{L^2} \\
&= ||[S_1 \mu_{N,\gamma} S_{-1} \mu_{N,\gamma}^{-1}][\mu_{N,\gamma} S_1 (\Delta + \lambda)^{-1} S_1 \mu_{N,\gamma}^{-1}][\mu_{N,\gamma} S_{-1} \mu_{N,\gamma}^{-1} S_1] f||_{L^2} \\
&\leq C_N ||f||_{L^2},
\end{aligned}$$

using Lemma 3.2 and (3.7). This proves (3.12).

The proof in the case $\lambda \in [\delta, \delta^{-1}]$ is more difficult, since the elliptic bound (3.12) does not hold. For some small constant $\varepsilon_0 = \varepsilon_0(\delta) > 0$ (to be fixed later), let $A = \{\xi^1, \ldots, \xi^m\}$ denote a $\varepsilon_0/100$-net on the sphere $\{|\xi| = \sqrt{\lambda}\}$ in the Fourier space. Using this net, we construct a partition of 1 in the Fourier space. We have $1 = \chi_0 + \chi_1 + \ldots + \chi_m$, where $\chi_0, \chi_1, \ldots, \chi_m : \mathbb{R}^d \to [0, 1]$ are smooth functions, χ_0 is supported in the set $\{\xi : |\sqrt{\lambda} - |\xi|| \geq \varepsilon_0/10\}$, and χ_j is supported in the set $\{\xi : |\xi^j - \xi| \leq \varepsilon_0/2\}$. Let $\chi_j(D)$ denote the operator defined by the Fourier multiplier χ_j.

An estimate similar to the proof of (3.12) shows that

$$\begin{aligned}||\mu_{N,\gamma}\chi_0(D)u||_{X^*} &\leq C||\mu_{N,\gamma}\chi_0(D)u||_{W^{1,2}} \leq C_{N,\delta}||\mu_{N,\gamma}\chi_0(D)(\Delta+\lambda)u||_{W^{-1,2}}\\ &\leq C_{N,\delta}||\mu_{N,\gamma}(\Delta+\lambda)u||_X,\end{aligned} \tag{3.13}$$

using (3.10). It remains to prove a similar estimate for $||\mu_{N,\gamma}\chi_j(D)u||_{X^*}$, $j = 1,\ldots,m$. Let $\widetilde{\chi}_j$ denote a smooth function supported in the set $\{\xi : |\xi^j - \xi| \leq \varepsilon_0\}$ and equal to 1 in the set $\{\xi : |\xi^j - \xi| \leq \varepsilon_0/2\}$. Thus $\chi_j\widetilde{\chi}_j = \chi_j$. Then

$$\begin{aligned}&||\mu_{N,\gamma}\chi_j(D)u||_{X^*} \leq ||S_{1/(d+1)}\mu_{N,\gamma}\chi_j(D)u||_{L^{p'_d}} + ||S_1\mu_{N,\gamma}\chi_j(D)u||_{B^*}\\ &= ||[S_{1/(d+1)}\mu_{N,\gamma}S_{-1/(d+1)}\mu_{N,\gamma}^{-1}][\mu_{N,\gamma}S_{1/(d+1)}\widetilde{\chi}_j(D)\mu_{N,\gamma}^{-1}]\mu_{N,\gamma}\chi_j(D)u||_{L^{p'_d}}\\ &+ ||[S_1\mu_{N,\gamma}S_{-1}\mu_{N,\gamma}^{-1}][\mu_{N,\gamma}S_1\widetilde{\chi}_j(D)\mu_{N,\gamma}^{-1}]\mu_{N,\gamma}\chi_j(D)u||_{B^*}\\ &\leq C_{N,\delta}||\mu_{N,\gamma}\chi_j(D)u||_{L^{p'_d}\cap B^*},\end{aligned} \tag{3.14}$$

using Lemma 3.2 and (3.7). A similar estimate, using again Lemma 3.2 and (3.7), together with decompositions as in the proof of (3.10), shows that

$$||\mu_{N,\gamma}\chi_j(D)(\Delta+\lambda)u||_{L^{p_d}+B} \leq C_{N,\delta}||\mu_{N,\gamma}\chi_j(D)(\Delta+\lambda)u||_X. \tag{3.15}$$

The estimates (3.10), (3.13), (3.14), and (3.15) show that it suffices to prove that for any $u \in \mathcal{S}(\mathbb{R}^d)$

$$||\mu_{N,\gamma}\chi_j(D)u||_{L^{p'_d}\cap B^*} \leq C_{N,\delta}||\mu_{N,\gamma}\chi_j(D)(\Delta+\lambda)u||_{L^{p_d}+B}, j = 1,\ldots,m. \tag{3.16}$$

It remains to prove (3.16). By rescaling and rotation invariance, we may assume that $\lambda = 1$ and $\chi_j = \chi$ is a smooth function supported in the ball of radius $\varepsilon_0/2$ around the unit vector $\xi^+ = (0,\ldots,0,1)$. Let $\xi_1^+,\ldots,\xi_d^+$ denote a basis of $\mathbb{R}^d$ of unit vectors in the ball $\{\xi : |\xi - \xi^+| \leq \varepsilon_0/2\}$. Clearly

$$|x| \leq C(|x\cdot\xi_1^+| + \ldots + |x\cdot\xi_d^+|).$$

It follows easily that

$$\mu_{N,\gamma}(x) \approx [\widetilde{\mu}_{N,\gamma}(|x\cdot\xi_1^+|) + \ldots + \widetilde{\mu}_{N,\gamma}(|x\cdot\xi_d^+|)],$$

where, for $t \in [0,\infty)$

$$\widetilde{\mu}_{N,\gamma}(t) = \frac{(1+t^2)^N}{(1+\gamma t^2)^N}.$$

Thus we may replace the weight $\mu_{N,\gamma}(x)$ in (3.16) with $\widetilde{\mu}_{N,\gamma}(|x\cdot\xi_l^+|)$ (in both sides of the inequality). To summarize, by rotation invariance, it remains to prove that

$$||\widetilde{\mu}_{N,\gamma}(|x_d|)u||_{L^{p'_d}\cap B^*} \leq C_N||\widetilde{\mu}_{N,\gamma}(|x_d|)(\Delta+1)u||_{L^{p_d}+B}, \tag{3.17}$$

for all functions $u \in \mathcal{S}(\mathbb{R}^d)$ with the property that $\hat{u}$ is supported in the ball $\{\xi : |\xi - \xi^+| \leq \varepsilon_0\}$.

Let $\mathbf{1}_+$ and $\mathbf{1}_-$ denote the characteristic functions of the intervals $[0,\infty)$ and $(-\infty,0)$. Let $f = (\Delta+1)u$, $F(x) = \widetilde{\mu}_{N,\gamma}(|x_d|)f(x)$, and $U(x) = \widetilde{\mu}_{N,\gamma}(|x_d|)u(x)$.

Let $\widetilde{u}(\xi', x_d)$ and $\widetilde{f}(\xi', x_d)$ denote the partial Fourier transforms of the functions u and f in the variable $x' = (x_1, \ldots, x_{d-1})$. The equation $f = (\Delta + 1)u$ is equivalent to

$$[\partial_{x_d}^2 + (1 - |\xi'|^2)]\widetilde{u}(\xi', x_d) = \widetilde{f}(\xi', x_d). \tag{3.18}$$

The functions $\widetilde{u}$ and $\widetilde{f}$ are supported in the ball $\{\xi' : |\xi'| \leq \varepsilon_0 \ll 1\}$. By integration by parts,

$$\begin{aligned}\widetilde{u}(\xi', x_d) &= -\int_{x_d}^{\infty} \widetilde{f}(\xi', y_d) \frac{\sin(\sqrt{1 - |\xi'|^2}(x_d - y_d))}{\sqrt{1 - |\xi'|^2}}\, dy_d \\ &= \int_{-\infty}^{x_d} \widetilde{f}(\xi', y_d) \frac{\sin(\sqrt{1 - |\xi'|^2}(x_d - y_d))}{\sqrt{1 - |\xi'|^2}}\, dy_d.\end{aligned} \tag{3.19}$$

We use the formula in the first line of (3.19) when $x_d \geq 0$, and the formula in the second line when $x_d \leq 0$. Let $\phi : \mathbb{R}^{d-1} \to [0, 1]$ denote a smooth function supported in the ball $\{\xi' : |\xi'| \leq 2\varepsilon_0\}$ and equal to 1 in the ball $\{\xi' : |\xi'| \leq \varepsilon_0\}$. By taking the inverse Fourier transform in the variable ξ' we have

$$\begin{aligned}u(x', x_d) = c\mathbf{1}_+(x_d) &\int_{\mathbb{R}^d} f(y', y_d)\mathbf{1}_-(x_d - y_d)H(x' - y', x_d - y_d)\, dy \\ - c\mathbf{1}_-(x_d) &\int_{\mathbb{R}^d} f(y', y_d)\mathbf{1}_+(x_d - y_d)H(x' - y', x_d - y_d)\, dy,\end{aligned} \tag{3.20}$$

where

$$H(z', z_d) = \int_{\mathbb{R}^{d-1}} e^{iz'\cdot\xi'} \frac{\sin(\sqrt{1 - |\xi'|^2} z_d)}{\sqrt{1 - |\xi'|^2}} \phi(\xi')\, d\xi'. \tag{3.21}$$

By multiplying with the weight $\widetilde{\mu}_{N,\gamma}$, we have

$$U(x', x_d) = c\int_{\mathbb{R}^d} F(y', y_d)K(x', x_d, y', y_d)\, dy'dy_d, \tag{3.22}$$

where

$$\begin{aligned}K(x', x_d, y', y_d) = [&\mathbf{1}_+(x_d)\mathbf{1}_-(x_d - y_d) - \mathbf{1}_-(x_d)\mathbf{1}_+(x_d - y_d)] \\ &\frac{\widetilde{\mu}_{N,\gamma}(|x_d|)}{\widetilde{\mu}_{N,\gamma}(|y_d|)} H(x' - y', x_d - y_d).\end{aligned} \tag{3.23}$$

It is important to notice that $K(x', x_d, y', y_d) = 0$ if $|y_d| > |x_d|$; therefore the weight $\widetilde{\mu}_{N,\gamma}(|x_d|)/\widetilde{\mu}_{N,\gamma}(|y_d|)$ is always ≤ 1. Let T denote the operator defined by the kernel K in the right-hand side of (3.22). It remains to prove that T extends to a bounded operator from $L^{p_d} + B$ to $L^{p_d'} \cap B^*$.

Lemma 3.3. *For $f \in \mathcal{S}(\mathbb{R}^d)$ we have*

$$||Tf||_{B^*} \leq C||f||_B.$$

Proof of Lemma 3.3. This is essentially proved in [6, Chapter XIV]. We need the observation that

$$||Tf||_{B^*} \leq C \sup_{x_d \in \mathbb{R}} ||Tf(.,x_d)||_{L^2_{x'}} \text{ and } ||f||_B \geq C^{-1} \int_{\mathbb{R}} ||f(.,y_d)||_{L^2_{y'}} \, dy_d.$$

In addition, if $x_d \geq 0$ then

$$\begin{aligned} ||Tf(.,x_d)||_{L^2_{x'}} &= || \int_{\mathbb{R}^d} f(y',y_d) \mathbf{1}_-(x_d - y_d) \frac{\widetilde{\mu}_{N,\gamma}(x_d)}{\widetilde{\mu}_{N,\gamma}(y_d)} H(x'-y', x_d - y_d) \, dy ||_{L^2_{x'}} \\ &\leq \int_{x_d}^{\infty} \frac{\widetilde{\mu}_{N,\gamma}(x_d)}{\widetilde{\mu}_{N,\gamma}(y_d)} || \int_{\mathbb{R}^{d-1}} f(y',y_d) H(x'-y', x_d - y_d) \, dy' ||_{L^2_{x'}} dy_d \\ &\leq C \int_{\mathbb{R}} ||f(.,y_d)||_{L^2_{y'}} \, dy_d, \end{aligned}$$

where the last inequality follows from Plancherel's Theorem and the monotonicity of the weight $\widetilde{\mu}_{N,\gamma}$. The estimate in the case $x_d < 0$ is similar. $\square$

Lemma 3.4. *For $f \in \mathcal{S}(\mathbb{R}^d)$ we have*

$$||Tf||_{L^{p_d'}} \leq C ||f||_{L^{p_d}}. \tag{3.24}$$

Proof of Lemma 3.4. Let

$$W_N(x_d,y_d) = [\mathbf{1}_+(x_d)\mathbf{1}_-(x_d - y_d) - \mathbf{1}_-(x_d)\mathbf{1}_+(x_d - y_d)] \frac{\widetilde{\mu}_{N,\gamma}(|x_d|)}{\widetilde{\mu}_{N,\gamma}(|y_d|)}$$

denote the weight in the definition (3.23) of the kernel K, $W_N(x_d,y_d) \in [-1,1]$. We use analytic interpolation. For $\sigma \in \mathbb{C}$, $\Re\sigma \in [-(d-1)/2, 1]$, let

$$K^\sigma(x',x_d,y',y_d) = e^{\sigma^2}(1-\sigma)(1+|x_d - y_d|)^{-\sigma} K(x',x_d,y',y_d),$$

and T^σ the operator defined by the kernel K^σ. By analytic interpolation, it suffices to prove that

$$||T^\sigma||_{L^1 \to L^\infty} \leq C \ \text{ if } \Re\sigma = -(d-1)/2, \tag{3.25}$$

and

$$||T^\sigma||_{L^2 \to L^2} \leq C \ \text{ if } \Re\sigma = 1. \tag{3.26}$$

The bound (3.25) follows easily since $|H(z',z_d)| \leq C(1+|z_d|)^{-(d-1)/2}$, by stationary phase arguments.

To prove (3.26), we take partial Fourier transforms in the variables y' and x'. Let $W_N^\sigma(x_d,y_d) = e^{\sigma^2}(1-\sigma)(1+|x_d-y_d|)^{-\sigma} W_N(x_d,y_d)$. An easy computation shows that

$$\widetilde{T^\sigma f}(\eta',x_d) = c \int_{\mathbb{R}} \widetilde{f}(\eta',y_d) W_N^\sigma(x_d,y_d) \frac{\sin[\sqrt{1-|\eta'|^2}(x_d - y_d)]}{\sqrt{1-|\eta'|^2}} \phi(\eta') \, dy_d. \tag{3.27}$$

Notice that

$$\sin[\sqrt{1-|\eta'|^2}(x_d-y_d)] = c[e^{i\sqrt{1-|\eta'|^2}(x_d-y_d)} - e^{-i\sqrt{1-|\eta'|^2}(x_d-y_d)}].$$

We substitute this into (3.27). Notice that the exponential factors $e^{\pm i\sqrt{1-|\eta'|^2}x_d}$ and $e^{\pm i\sqrt{1-|\eta'|^2}y_d}$ can be paired with $\widetilde{T^\sigma f}(\eta', x_d)$ and $\widetilde{f}(\eta', y_d)$ respectively. By Plancherel's theorem, the L^2 bound (3.26) would follow once we prove that the kernel W_N^σ defines a bounded operator on $L^2(\mathbb{R})$:

$$||\int_{\mathbb{R}} h(y_d) W_N^\sigma(x_d, y_d)\, dy_d||_{L^2_{x_d}} \leq C||h||_{L^2_{y_d}} \text{ if } \Re\sigma = 1. \tag{3.28}$$

We will use the maximal operator

$$Mh(t) = \sup_{r\in\mathbb{R}} \left| \int_0^r h(t-s) e^{\sigma^2}(1-\sigma)(1+|s|)^{-\sigma}\, ds \right|.$$

For $h \in \mathcal{S}(\mathbb{R})$ and $x_d \geq 0$ we have

$$\begin{aligned}
|\int_{\mathbb{R}} h(y_d) W_N^\sigma(x_d, y_d)\, dy_d| &= |\int_{x_d}^\infty h(y_d) e^{\sigma^2}(1-\sigma)(1+|x_d-y_d|)^{-\sigma} \frac{\widetilde{\mu}_{N,\gamma}(x_d)}{\widetilde{\mu}_{N,\gamma}(y_d)}\, dy_d| \\
&= |\int_{x_d}^\infty \frac{d}{dy_d} \left(\int_{x_d}^{y_d} h(s) e^{\sigma^2}(1-\sigma)(1+|x_d-s|)^{-\sigma}\, ds \right) \frac{\widetilde{\mu}_{N,\gamma}(x_d)}{\widetilde{\mu}_{N,\gamma}(y_d)}\, dy_d| \\
&= |\int_{x_d}^\infty \left(\int_{x_d}^{y_d} h(s) e^{\sigma^2}(1-\sigma)(1+|x_d-s|)^{-\sigma}\, ds \right) \frac{d}{dy_d} \frac{\widetilde{\mu}_{N,\gamma}(x_d)}{\widetilde{\mu}_{N,\gamma}(y_d)}\, dy_d| \\
&\leq \int_{x_d}^\infty Mh(x_d) \left| \frac{d}{dy_d} \frac{\widetilde{\mu}_{N,\gamma}(x_d)}{\widetilde{\mu}_{N,\gamma}(y_d)} \right| dy_d \leq Mh(x_d).
\end{aligned} \tag{3.29}$$

The last inequality is due to the fact that the function $y_d \to [\mu_{N,\gamma}(x_d)/\mu_{N,\gamma}(y_d)]$ is nonincreasing, thus it has bounded variation. A similar computation proves the estimate (3.29) in the case $x_d < 0$. In addition, when $\Re\sigma = 1$, the kernels $\chi_\pm(s) e^{\sigma^2}(1-\sigma)(1+|s|)^{-\sigma}$ are Calderón–Zygmund kernels, uniformly in σ. Therefore the maximal operator M is bounded on $L^2(\mathbb{R})$ (see, for example, [18, Chapter I, Section 7], so the bound (3.28) follows from (3.29). □

Lemma 3.5. *For $f \in \mathcal{S}(\mathbb{R}^d)$ we have*

$$||T||_{L^{p_d} \to B^*} + ||T||_{B \to L^{p'_d}} \leq C. \tag{3.30}$$

Proof of Lemma 3.5. We prove the bound for the first term in (3.30) (the proof for the second term is identical). As in the proof of Lemma 3.3, it suffices to prove that

$$||Tf(., x_d)||_{L^2_{x'}} \leq C||f||_{L^{p_d}}. \tag{3.31}$$

Assuming x_d fixed, let $g(y', y_d) = f(y', y_d)W_N(x_d, y_d)$, where $W_N(x_d, y_d) \in [-1, 1]$ is the weight defined in the proof of Lemma 3.4. Clearly, $||g||_{L^{p_d}} \leq ||f||_{L^{p_d}}$. Also,

$$Tf(x',x_d) = \int_{\mathbb{R}^d} g(y', y_d)H(x'-y', x_d-y_d)\,dy'dy_d = \frac{1}{2i}\int_{\mathbb{R}^{d-1}} e^{ix'\cdot\xi'}\frac{\phi(\xi')}{\sqrt{1-|\xi'|^2}}$$
$$[\widehat{g}(\xi', \sqrt{1-|\xi'|^2})e^{i\sqrt{1-|\xi'|^2}x_d} - \widehat{g}(\xi', -\sqrt{1-|\xi'|^2})e^{-i\sqrt{1-|\xi'|^2}x_d}]\,d\xi'.$$

The bound (3.31) follows from Plancherel theorem and the Stein-Tomas restriction theorem. □

We remove now the restriction (3.11). Let $\varphi : \mathbb{R}^d \to [0, C]$ denote a smooth function supported in the ball $\{x : |x| \leq 1\}$ with $\int_{\mathbb{R}^d}\varphi\,dx = 1$, and $\chi : \mathbb{R}^d \to [0, 1]$ a smooth function supported in the ball $\{x : |x| \leq 2\}$ and equal to 1 in the ball $\{x : |x| \leq 1\}$. For $\varepsilon \in (0, 1]$ and $r \in [1,\infty)$, let $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$ and $\chi_r(x) = \chi(x/r)$. Let

$$u_{\varepsilon,r}(x) = \chi_r(x)(u * \varphi_\varepsilon)(x).$$

Clearly $u_{\varepsilon,r} \in \mathcal{S}(\mathbb{R}^d)$ and

$$(\Delta + \lambda)u_{\varepsilon,r} = \chi_r[(\Delta + \lambda)u * \varphi_\varepsilon] + 2\nabla\chi_r \cdot \nabla(u * \varphi_\varepsilon) + \Delta\chi_r(u * \varphi_\varepsilon).$$

We apply Theorem 1.2 to the Schwartz function $u_{\varepsilon,r}$. The result is

$$\begin{aligned} ||\mu_{N,\gamma}\chi_r(u * \varphi_\varepsilon)||_{X^*} &\leq C_{N,\delta}||\mu_{N,\gamma}\chi_r[(\Delta + \lambda)u * \varphi_\varepsilon]||_{X_q} \\ &\quad + C_{N,\delta}[||\mu_{N,\gamma}\nabla\chi_r \cdot \nabla(u * \varphi_\varepsilon)||_B + ||\mu_{N,\gamma}\Delta\chi_r(u * \varphi_\varepsilon)||_B]. \end{aligned} \tag{3.32}$$

The function $\nabla\chi_r$ and $\Delta\chi_r$ are both supported in the set $\{x : |x| \in [r, 2r]\}$ and dominated by C/r. In addition, $|\nabla(u * \varphi_\varepsilon)| \leq C\varepsilon^{-1}|u| * (\varepsilon^{-d}|\nabla\varphi(./\varepsilon)|)$. By (1.10), assuming ε fixed and letting $r \to \infty$ in (3.32), we have

$$||\mu_{N,\gamma}(u * \varphi_\varepsilon)||_{X^*} \leq C_{N,\delta}||\mu_{N,\gamma}(\Delta + \lambda)v * \varphi_\varepsilon||_X.$$

The theorem follows by letting $\varepsilon \to 0$.

4. The operators $\mathrm{Id}_{X^*} + R_0(\lambda \pm i\epsilon)L$

Following the classical scheme, we will transfer some of the previous estimates for the free resolvent to the perturbed resolvent by means of the resolvent identity. This requires inverting

$$\mathrm{Id}_{X^*} + R_0(\lambda \pm i0)L \tag{4.1}$$

as an operator on X^*. We start with the definition of the operators $R_0(\lambda \pm i0)$.

Lemma 4.1. *(a) The map $z \to R_0(z)$ defines an analytic map from $\mathbb{C} \setminus [0,\infty)$ to $\mathcal{L}(X, X^*)$.*

(b) For any $\lambda \in (0,\infty)$ there are operators $R_0(\lambda+i0), R_0(\lambda-i0) \in \mathcal{L}(X,X^*)$ with the property that

$$||R_0(\lambda \pm i0)||_{X\to X^*} \leq C_\delta \text{ for any } \lambda \in [\delta,\delta^{-1}], \delta > 0. \tag{4.2}$$

In addition, for any sequences $\{\lambda_n\}_{n=1}^\infty \subseteq (0,\infty)$ and $\{\epsilon_n\}_{n=1}^\infty \subseteq [0,\infty)$ with $\lambda_n \to \lambda$ and $\epsilon_n \to 0$, we have

$$\lim_{n\to\infty} \langle R_0(\lambda_n \pm i\epsilon_n)f, \phi\rangle = \langle R_0(\lambda \pm i0)f, \phi\rangle \tag{4.3}$$

for any $f \in X$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$, and

$$\lim_{n\to\infty} ||\mathbf{1}_{\{|x|\leq R\}}[R_0(\lambda_n \pm i\epsilon_n)f - R_0(\lambda \pm i0)f]||_{L^2} = 0 \tag{4.4}$$

for any $f \in X$ and $R \geq 1$.

(c) For $\lambda \in \mathbb{R}\setminus\{0\}$, $\epsilon \geq 0$, and $g \in X$

$$[-\Delta - (\lambda \pm i\epsilon)]R_0(\lambda \pm i\epsilon)g = g$$

in the sense of distributions (by a slight abuse of notation we let $R_0(\lambda+i0) = R_0(\lambda-i0) := R_0(\lambda)$ when $\lambda \in (-\infty,0)$).

Proof of Lemma 4.1. Part (a) follows directly from the definitions; in fact, the map $z \to R_0(z)$ defines an analytic map from $\mathbb{C}\setminus[0,\infty)$ to $\mathcal{L}(W^{-1,2},W^{1,2})$.

For part (b), we use the fact that $R_0(z)f = f * R_z$ for $z \in \mathbb{C}\setminus[0,\infty)$ and $f \in \mathcal{S}(\mathbb{R}^d)$, where

$$R_z(x) = C(z^{1/2}/|x|)^{(d-2)/2}K_{(d-2)/2}(-iz^{1/2}|x|). \tag{4.5}$$

Here K_ν denote the Bessel potentials and, as before, $\Im(z^{1/2}) > 0$ (see [4, p. 288]). Standard estimates on the Bessel potentials show that if $|z| \in [\delta,\delta^{-1}]$ then

$$|R_z(x)| \leq C_\delta \begin{cases} |x|^{-(d-1)/2} & \text{if } |x| \geq 1; \\ |x|^{-(d-2)} & \text{if } |x| \leq 1 \text{ and } d \geq 3; \\ \log(2/|x|) & \text{if } |x| \leq 1 \text{ and } d = 2. \end{cases} \tag{4.6}$$

We define the kernels $R_{\lambda+i0}(x)$ and $R_{\lambda-i0}(x)$ using the formula (4.5) and letting $z \to \lambda + i0$ and $z \to \lambda - i0$. The kernels $R_{\lambda\pm i0}(x)$ satisfy the bound (4.6). Then, for $f \in \mathcal{S}(\mathbb{R}^d)$, we define

$$R_0(\lambda \pm i0)f := f * R_{\lambda\pm i0}.$$

Using the Lebesgue dominated convergence theorem and (4.6),

$$\lim_{n\to\infty} f * R_{\lambda_n \pm i\epsilon_n}(x) = f * R_{\lambda\pm i0}(x) \tag{4.7}$$

for any $f \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, where λ_n and ϵ_n are as in part (b). Using the Fatou lemma and Theorem 1.1, for any $f \in \mathcal{S}(\mathbb{R}^d)$ and $\lambda \in [\delta, \delta^{-1}]$,

$$||R_0(\lambda \pm i0)f||_{S_{-1}(B^*)} = ||(S_1 f) * R_{\lambda \pm i0}||_{B^*} \leq \limsup_{n\to\infty} ||(S_1 f) * R_{\lambda \pm i/n}||_{B^*}$$
$$\leq \limsup_{n\to\infty} ||f * R_{\lambda\pm i/n}||_{S_{-1}(B^*)} \leq C_\delta ||f||_X.$$

A similar estimate shows that $||R_0(\lambda \pm i0)f||_{W^{1/(d+1),p'_d}} \leq C_\delta ||f||_X$ for any $f \in \mathcal{S}(\mathbb{R}^d)$. Thus the operators $R_0(\lambda \pm i0)$ extend to bounded operators from X to X^* and (4.2) holds.

To prove the limits (4.3) and (4.4), we notice that we may assume $f \in \mathcal{S}(\mathbb{R}^d)$, in view of (4.2) and the fact that $\mathcal{S}(\mathbb{R}^d)$ is dense in X. The limits (4.3) and (4.4) then follow from the Lebesgue dominated convergence theorem, the poinwise limit (4.7), and the observation that $|f * R_{\lambda_n \pm i\epsilon_n}(x)| \leq C||f||_{\mathcal{S}(\mathbb{R}^d)}$ (using (4.6)).

For part (c) we have to prove that for $g \in X$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle [-\Delta - (\lambda \pm i\epsilon)] R_0(\lambda \pm i\epsilon) g, \phi\rangle = \langle g, \phi\rangle,$$

which is equivalent to

$$\langle R_0(\lambda \pm i\epsilon) g, [-\Delta - (\lambda \mp i\epsilon)]\phi\rangle = \langle g, \phi\rangle. \tag{4.8}$$

When $\lambda < 0$ or $\epsilon \neq 0$, the identity (4.8) is clear, for any $g \in \mathcal{S}'(\mathbb{R}^d)$. When $\lambda > 0$ and $\epsilon = 0$, the identity (4.8) follows from (4.3). $\square$

Next we establish the compactness of the operator L.

Lemma 4.2. *If L is an admissible perturbation, then $L : X^* \to X$ is a compact operator.*

Proof of Lemma 4.2. Let $\{f_n\}_{n=1}^\infty$ denote a sequence of functions in X^* with $||f_n||_{X^*} \leq 1$. Let $\chi : \mathbb{R}^d \to [0, 1]$ denote a smooth function supported in the set $\{|x| \leq 2\}$ and equal to 1 in the set $\{|x| \leq 1\}$. For any $r \geq 1$ let $\chi_r(x) = \chi(x/r)$. Since $||S_1(f_n)||_{B^*} \leq 1$,

$$||f_n \chi_r||_{W^{1,2}} \leq C_r \tag{4.9}$$

for any $n \geq 1$ and $r \geq 1$.

We use first (4.9) with $r = 1$. By the Rellich–Kondrachov compactness theorem, there is a subsequence $\{f_{1,n}\}_{n=1}^\infty \subseteq \{f_n\}_{n=1}^\infty$ and a function $g_1 \in L^2$ with the property that

$$\lim_{n\to\infty} f_{1,n}\chi_1 = g_1 \text{ in } L^2.$$

We repeat this argument inductively for $r = 2, 3, \ldots$ and construct subsequences $\{f_{k,n}\}_{n=1}^\infty \subseteq \{f_{k-1,n}\}_{n=1}^\infty$ and functions $g_k \in L^2$ with the property that

$$\lim_{n\to\infty} f_{k,n}\chi_k = g_k \text{ in } L^2. \tag{4.10}$$

We consider the diagonal subsequence $\widetilde{f}_k := f_{k,k}$, $k = 1, 2, \ldots$. It remains to prove that $L\widetilde{f}_k$ is a Cauchy sequence in X. Given $\varepsilon > 0$, we use (1.14) with $N = 0$. Therefore, there are constants A_ε and R_ε with the property that

$$||L(\widetilde{f}_k - \widetilde{f}_{k'})||_X \le (\varepsilon/4)||\widetilde{f}_k - \widetilde{f}_{k'}||_{X^*} + A_\varepsilon||(\widetilde{f}_k - \widetilde{f}_{k'})\chi_{R\varepsilon}||_{L^2}.$$

By (4.10) and the definition of $\widetilde{f}_k$

$$\limsup_{k,k'\to\infty} ||(\widetilde{f}_k - \widetilde{f}_{k'})\chi_{R\varepsilon}||_{L^2} = 0.$$

Thus $||L(\widetilde{f}_k - \widetilde{f}_{k'})||_X \le \varepsilon$ for k, k' large enough, as desired. □

The following is a technical lemma which will be needed in the proof of invertibility of $\mathrm{Id}_{X^*} + R_0(\lambda \pm i\epsilon)L$.

Lemma 4.3. *Assume $\phi \in C_0^\infty(\mathbb{R}^d)$, $\phi(0) = 1$, and $\phi(x) = 0$ if $|x| \ge 1$. Then for any $\lambda > 0$ and $g \in X$,*

$$\Im\langle g, R_0(\lambda \pm i0)g\rangle = c_1 \int_{\sqrt{\lambda}S^{d-1}} |\hat{g}(\xi)|^2\, \sigma(d\xi)$$

$$\lim_{R\to\infty} \int_{\mathbb{R}^d} |R_0(\lambda \pm i0)g|^2(x)\, \phi\Big(\frac{x}{R}\Big)\, \frac{dx}{R} = c_2 \int_{\sqrt{\lambda}S^{d-1}} |\hat{g}(\xi)|^2\, \sigma(d\xi)$$

where $c_1 = c_1(\lambda, \pm) \neq 0$ and $c_2 = c_2(\lambda, \pm, \phi) \neq 0$.

Proof of Lemma 4.3. If $g \in \mathcal{S}(\mathbb{R}^d)$, then these properties are standard, see for example [6, Chapter XIV]. Clearly, $\mathcal{S}(\mathbb{R}^d)$ is dense in X. Moreover, in view of Theorem 1.1 the left-hand sides of these limits are continuous with respect to the norm of X. Finally, by the trace lemma and the Stein-Tomas theorem, respectively, the right-hand sides are also continuous with respect to the X-norm, which proves the identities. □

Assume from now on that L is the admissible perturbation in Theorem 1.3. We define a set $\widetilde{\mathcal{E}} \subset \mathbb{R}\setminus\{0\}$ so that off this set (4.1) is invertible. We will show later that $\widetilde{\mathcal{E}}$ is exactly the set of nonzero eigenvalues, which we denoted by $\mathcal{E}$ in Theorem 1.3. Let

$$\widetilde{\mathcal{E}}^\pm := \{\lambda \in \mathbb{R}\setminus\{0\} : \text{there is } f \in X^*\setminus\{0\} \text{ with } (\mathrm{Id}_{X^*} + R_0(\lambda \pm i0)L)f = 0\}. \tag{4.11}$$

Notice that $R_0(\lambda - i0)\overline{g} = \overline{R_0(\lambda + i0)g}$ for any $g \in X$. Thus $\widetilde{\mathcal{E}}^+ = \widetilde{\mathcal{E}}^- := \widetilde{\mathcal{E}}$. For any $\lambda \in \mathbb{R}\setminus\{0\}$ we define the eigenspaces

$$\mathcal{F}_\lambda^\pm = \{f \in X^* : (\mathrm{Id}_{X^*} + R_0(\lambda \pm i0)L)f = 0\}.$$

Lemma 4.4. *Assume that $\lambda \in \widetilde{\mathcal{E}}$ and $f \in \mathcal{F}_\lambda^\pm$. Then $(1 + |x|^2)^N f \in X^*$ for any $N \ge 0$, and*

$$||(1 + |x|^2)^N f||_{X^*} \le C_{N,L,\lambda}\|f\|_{X^*}. \tag{4.12}$$

Proof of Lemma 4.4. We show first that

$$\langle Lf, g\rangle = \overline{\langle Lg, f\rangle} \tag{4.13}$$

for any $f, g \in X^*$ (this is assumed in (1.13) for $f, g \in \mathcal{S}(\mathbb{R}^d)$). Using the functions χ and φ defined at the end of section 3 we define the sequences

$$f_n(x) = \chi_n(x)(f * \varphi_{1/n})(x) \text{ and } g_n(x) = \chi_n(x)(g * \varphi_{1/n})(x).$$

Clearly, $f_n, g_n \in \mathcal{S}(\mathbb{R}^d)$, $||f_n||_{X^*} \le C||f||_{X^*}$, and $||g_n||_{X^*} \le C||g||_{X^*}$. In view of (1.13), it suffices to prove that

$$\lim_{n\to\infty} \langle Lf_n, g_n\rangle = \langle Lf, g\rangle. \tag{4.14}$$

We remark that the sequences f_n and g_n may not converge to f and g in X^* (in fact $\mathcal{S}(\mathbb{R}^d)$ is not dense in X^*). However, using (1.11) and the fact that $\mathcal{S}(\mathbb{R}^d)$ is dense in X, for the limit above it suffices to prove that

$$\lim_{n\to\infty} ||L(f_n - f)||_X = 0, \tag{4.15}$$

and

$$\lim_{n\to\infty} \langle (g_n - g), \phi\rangle = 0 \text{ for any } \phi \in \mathcal{S}(\mathbb{R}^d). \tag{4.16}$$

The limit (4.16) is clear, even for any $g \in L^{p'_d}$. For the limit (4.15), given $\varepsilon > 0$, we use (1.14) with $N = 0$. The result is

$$||L(f_n - f)||_X \le \varepsilon||f_n - f||_{X^*} + A_\varepsilon||f_n - f||_{L^2(\{|x|\le R_\varepsilon\})}.$$

Since $\lim_{n\to\infty} ||f_n - f||_{L^2(\{|x|\le R\})} = 0$ for any $R \ge 1$, the limit (4.15) follows.

Assume that

$$f + R_0(\lambda \pm i0)Lf = 0 \tag{4.17}$$

for some $f \in X^*$, $\lambda \in \mathbb{R} \setminus \{0\}$, and some choice of $+$ or $-$. If $\lambda > 0$, then we use Lemma 4.3 and the fact that $\langle Lf, f\rangle \in \mathbb{R}$ to conclude that

$$0 = \Im\langle Lf, R_0(\lambda \pm i0)Lf\rangle = c\int_{\sqrt{\lambda}S^{d-1}} |\widehat{Lf}|^2\, d\sigma$$

with some constant $c \ne 0$. Applying Lemma 4.3 again implies that

$$\lim_{R\to\infty} R^{-1}\int_{\{|x|\le R\}} \Big|[R_0(\lambda \pm i0)Lf](x)\Big|^2\, dx = 0.$$

In view of (4.17) this is the same as

$$\lim_{R\to\infty} R^{-1}\int_{\{|x|\le R\}} |f(x)|^2\, dx = 0. \tag{4.18}$$

Let $\delta > 0$ be such that $\delta \le |\lambda| \le \delta^{-1}$. By Lemma 4.1 and Theorem 1.2

$$||\mu_{N,\gamma}f||_{X^*} \le C_{N,\delta}||\mu_{N,\gamma}Lf||_X.$$

We use (1.14) with $\varepsilon = (2C_{N,\delta})^{-1}$ and the fact that $||\mu_{N,\gamma} f||_{X^*} < \infty$. By absorbing the term $(1/2)||\mu_{N,\gamma} f||_{X^*}$,

$$||\mu_{N,\gamma} f||_{X^*} \le C_{N,L,\delta} ||f||_{B^*}.$$

The inequality (4.12) follows by letting $\gamma \to 0$.

If $\lambda < 0$, then since $Lf \in X \hookrightarrow W^{-1,2}$, we have $R_0(\lambda)Lf \in W^{1,2}$, thus (4.18) holds, using (4.17). The same argument as above proves (4.12). $\square$

Lemma 4.5. *(a) For any* $\lambda \in \mathbb{R} \setminus \{0\}$

$$\mathcal{F}^{\pm}_{\lambda} \subseteq \{u \in \mathrm{Domain}(H) : Hu = \lambda u\}.$$

In particular, $\widetilde{\mathcal{E}} \subseteq \mathcal{E}$.

(b) The set $\widetilde{\mathcal{E}}$ *is discrete in* $\mathbb{R} \setminus \{0\}$*, i.e.,* $I \cap \widetilde{\mathcal{E}}$ *is finite for any compact set* $I \subseteq \mathbb{R} \setminus \{0\}$.

(c) For any $\lambda \in \mathbb{R} \setminus \{0\}$*, the vector spaces* $\mathcal{F}^{\pm}_{\lambda}$ *are finite-dimensional.*

Proof of Lemma 4.5. For part (a), by Lemma 4.1 (c)

$$(-\Delta - \lambda) f + Lf = 0 \tag{4.19}$$

for any $f \in \mathcal{F}^{\pm}_{\lambda}$. By Lemma 4.4, $f \in W^{1,2}$, thus $f \in$ Domain(H).

We prove now part (b). Assume, for contradiction, that the set $\widetilde{\mathcal{E}} \cap \{\lambda : \delta \le |\lambda| \le \delta^{-1}\}$ is infinite for some $\delta > 0$, thus $\widetilde{\mathcal{E}} \cap \{\lambda : \delta \le |\lambda| \le \delta^{-1}\} = \{\lambda_1, \lambda_2, \dots\}$, $\lambda_m \ne \lambda_n$ if $m \ne n$. For any n fix $f_n \in \mathcal{F}^{+}_{\lambda_n} \setminus \{0\}$. By (4.19), $f_m \ne f_n$ if $m \ne n$. By (4.12), $f_n \in W^{1,2}$. We normalize the functions f_n in such a way that $||f_n||_{W^{1,2}} = 1$. Then, by (4.12),

$$||(1 + |x|^2) f_n||_{W^{1,2}} \le C_{L,\delta} ||f_n||_{X^*} \le C_{L,\delta} \tag{4.20}$$

for any integer $n \ge 1$. Also, by (4.19), $(-\Delta + 1) f_n = (\lambda_n + 1 - L) f_n$, thus

$$\begin{aligned} f_n &= R_0(-1)[(\lambda_n + 1 - L) f_n] \\ &= (\lambda_n + 1) R_0(-1)(1 + |x|^2)^{-1}[(1 + |x|^2) f_n] - R_0(-1) L f_n. \end{aligned} \tag{4.21}$$

Using Lemma 4.2, it is easy to see that the operators

$$R_0(-1)(1 + |x|^2)^{-1}, R_0(-1)L : W^{1,2} \to W^{1,2} \text{ as compact operators.} \tag{4.22}$$

By (4.20) and (4.21), we pass to a subsequence and assume that $\lim_{n\to\infty} f_n = f_\infty$ in $W^{1,2}$. By the normalization of the functions f_n, $||f_\infty||_{W^{1,2}} = 1$. On the other hand, by part (a), the functions f_n are eigenfunctions of the self-adjoint operator H (see section 5) with different eigenvalues, thus $\langle f_m, f_n \rangle = 0$ if $m \ne n$. Therefore $f_\infty = 0$, which yields a contradiction.

For part (c), assume for contradiction that $\dim(\mathcal{F}^{\pm}_{\lambda}) = \infty$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Then we could find an infinite sequence of functions $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{F}^{\pm}_{\lambda}$, such that $||f_n||_{W^{1,2}} = 1$ and $\langle f_m, f_n \rangle = 0$ if $m \ne n$. The same argument as above gives a contradiction. $\square$

Finally, we need to prove that the operators $\mathrm{Id}_{X^*}+R_0(\lambda\pm i\epsilon)L$ are uniformly invertible on X^*, provided that λ is separated from $\widetilde{\mathcal{E}}$.

Lemma 4.6. *(a) For any $\lambda \in (\mathbb{R}\setminus\{0\})\setminus\widetilde{\mathcal{E}}$ and $\epsilon\in[0,\infty)$, the operator $\mathrm{Id}_{X^*}+R_0(\lambda\pm i\epsilon)L$ is invertible on X^*.*

(b) For any compact set $I\subseteq(\mathbb{R}\setminus\{0\})\setminus\widetilde{\mathcal{E}}$,

$$\sup_{\lambda\in I}\sup_{1\geq\epsilon\geq 0}\left\|(\mathrm{Id}_{X^*}+R_0(\lambda\pm i\epsilon)L)^{-1}\right\|_{X^*\to X^*}<\infty. \tag{4.23}$$

Proof of Lemma 4.6. For part (a) we use Lemma 4.2. Since $R_0(\lambda\pm i\epsilon)L$ is compact on X^*, the only alternative to invertibility is the existence of a nontrivial kernel. By the definition of the set $\widetilde{E}$, such a nontrivial kernel could only exist if $\epsilon>0$. If $f\in X^*$ has the property that

$$f+R_0(\lambda\pm i\epsilon)Lf=0, \tag{4.24}$$

then

$$\langle Lf,f\rangle+\langle Lf,R_0(\lambda+i\epsilon)Lf\rangle=0.$$

Since L is symmetric, by taking the imaginary part we have

$$0=\Im\langle Lf,R_0(\lambda+i\epsilon)Lf\rangle=\epsilon\int_{\mathbb{R}^d}|\widehat{Lf}|^2[(|\xi|^2-\lambda)^2+\epsilon^2]^{-1}\,d\xi.$$

Since $\epsilon\neq 0$, it follows that $Lf\equiv 0$, thus $f\equiv 0$ by (4.24).

For part (b), we show that

$$\sup_{\lambda\in I}\sup_{1\geq\epsilon\geq 0}\left\|(\mathrm{Id}_{X^*}+R_0(\lambda+i\epsilon)L)^{-1}\right\|_{X^*\to X^*}<\infty. \tag{4.25}$$

The proof for the operators $(\mathrm{Id}_{X^*}+R_0(\lambda-i\epsilon)L)^{-1}$ is identical. Assume for contradiction that the supremum on the left-hand side of (4.25) is ∞. Then there exist $f_n\in X^*$, $\|f_n\|_{X^*}=1$, such that

$$\|(\mathrm{Id}_{X^*}+R_0(\lambda_n+i\epsilon_n)L)f_n\|_{X^*}\to 0$$

as $n\to\infty$. Here $\lambda_n\in I$ and $\epsilon_n\in[0,1]$. We start from the identity

$$f_n=-R_0(\lambda_n+i\epsilon_n)Lf_n+r_n, \tag{4.26}$$

where $||r_n||_{X^*}\to 0$ as $n\to\infty$. By passing to a subsequence, we may assume that $\lambda_n+i\epsilon_n\to\lambda_\infty+i\epsilon_\infty\in I\times[0,1]$ and, since $L:X^*\to X$ is compact (Lemma 4.2), $Lf_n\to h$ in X. Let $f_\infty=-R_0(\lambda_\infty+i\epsilon_\infty)h\in X^*$. Using (4.26), for any $\phi\in\mathcal{S}(\mathbb{R}^d)$ we have

$$\begin{aligned}\lim_{n\to\infty}\langle f_n,\phi\rangle&=\lim_{n\to\infty}\langle -R_0(\lambda_n+i\epsilon_n)Lf_n,\phi\rangle\\&=\lim_{n\to\infty}\langle -R_0(\lambda_n+i\epsilon_n)h,\phi\rangle+\lim_{n\to\infty}\langle R_0(\lambda_n+i\epsilon_n)(h-Lf_n),\phi\rangle=\langle f_\infty,\phi\rangle.\end{aligned} \tag{4.27}$$

In the last identity we used Lemma 4.1 and Theorem 1.1. In addition, for any $\varepsilon > 0$ we use (1.14) with $N = 0$:

$$\begin{aligned}||Lf_n - Lf_\infty||_X &\le \varepsilon||f_n - f_\infty||_{X^*} + A_\varepsilon||(f_n - f_\infty)\mathbf{1}_{\{|x|\le R_\varepsilon\}}||_{L^2}\\ &\le \varepsilon||f_n - f_\infty||_{X^*} + C_\varepsilon[||r_n||_{X^*} + ||R_0(\lambda_n + i\epsilon_n)(Lf_n - h)||_{X^*}]\\ &+ A_\varepsilon||\mathbf{1}_{\{|x|\le R_\varepsilon\}}[R_0(\lambda_n + i\epsilon_n)h - R_0(\lambda_\infty + i\epsilon_\infty)h]||_{L^2}.\end{aligned}$$

By Lemma 4.1 and Theorem 1.1

$$\lim_{n\to\infty} Lf_n = Lf_\infty \text{ in } X. \tag{4.28}$$

It follows from (4.27), (4.28) and Lemma 4.1 that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$

$$0 = \lim_{n\to\infty}\langle f_n + R_0(\lambda_n + i\epsilon_n)Lf_n, \phi\rangle = \langle f_\infty + R_0(\lambda_\infty + i\epsilon_\infty)Lf_\infty, \phi\rangle.$$

Thus $f_\infty + R_0(\lambda_\infty + i\epsilon_\infty)Lf_\infty = 0$, which, in view of part (a), shows that $f_\infty = 0$. By (4.28), $\lim_{n\to\infty} Lf_n = 0$ in X. This gives a contradiction, in view of the identity (4.26) and the fact that $||f_n||_{X^*} = 1$. □

5. Proof of Theorem 1.3

We can now finish the proof of Theorem 1.3.

Proof of part (a): We show first that if $u \in \text{Domain}(H)$ then

$$||u||^2_{W^{1,2}} \le C_L||u||^2_{L^2} + C_0\langle Hu, u\rangle, \tag{5.1}$$

for some constants $C_0 \ge 0$ and C_L. To see this, we start from the identity

$$\langle Hu, \phi\rangle = \langle\nabla u, \nabla\phi\rangle + \langle Lu, \phi\rangle,$$

valid, by definition, for any $u \in \text{Domain}(H)$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$. Since $u \in \text{Domain}(H)$, and $\mathcal{S}(\mathbb{R}^d)$ is dense in $W^{1,2}(\mathbb{R}^d)$, we conclude that

$$\langle Hu, v\rangle = \langle\nabla u, \nabla v\rangle + \langle Lu, v\rangle, \tag{5.2}$$

for any $u, v \in \text{Domain}(H)$. Here we used that $L : W^{1,2} \to W^{-1,2}$. In particular, $\langle Hu, u\rangle \in \mathbb{R}$ for any $u \in \text{Domain}(H)$. For $\varepsilon > 0$ small enough, we use (1.14) with $N = 0$. The result is

$$\begin{aligned}\langle Hu, u\rangle &\ge ||\,|\nabla u|\,||^2_{L^2} - |\langle Lu, u\rangle|\\ &\ge c_0||u||^2_{W^{1,2}} - C||u||^2_{L^2} - ||u||_{X^*}(\varepsilon||u||_{X^*} + A_\varepsilon||u||_{L^2})\\ &\ge c_0||u||^2_{W^{1,2}} - C||u||^2_{L^2} - C\varepsilon||u||^2_{W^{1,2}} - C_\varepsilon||u||^2_{L^2}\\ &\ge (c_0/2)||u||^2_{W^{1,2}} - C_L||u||^2_{L^2},\end{aligned}$$

by choosing ε small enough. This proves (5.1).

An elementary limiting argument, using (5.1) and the fact that $W^{1,2}$ is a Banach space, shows that the operator

$$H : \text{Domain}(H) \to L^2$$

is closed. Clearly, it is also bounded from below. Finally, using (5.2) and the fact that L is symmetric, the operator H is symmetric. It remains to prove that Domain(H) is dense in L^2, and, using the criterion for self-adjointness [14, Theorem VIII.3], that

$$\mathrm{Range}(H \pm i) = L^2. \tag{5.3}$$

Lemma 5.1. *If $\lambda \in \mathbb{R}$, and $\epsilon \in \mathbb{R} \setminus \{0\}$, then*

$$\widetilde{R}_L(\lambda + i\epsilon) := (\mathrm{Id}_{W^{1,2}} + R_0(\lambda + i\epsilon)L)^{-1} R_0(\lambda + i\epsilon)$$

defines a bounded operator $\widetilde{R}_L(\lambda + i\epsilon) : L^2 \to \mathrm{Domain}(H)$. In addition,

$$[H - (\lambda + i\epsilon)]\widetilde{R}_L(\lambda + i\epsilon) = \mathrm{Id}_{L^2}. \tag{5.4}$$

Assuming Lemma 5.1, the density of Domain(H) in L^2 and (5.3) follow easily (for the density of Domain(H), notice that Domain(H) contains the set $(\mathrm{Id}_{W^{1,2}} + R_0(i)L)^{-1}W^{2,2}$, which is dense in $W^{1,2}$, thus dense in L^2). In addition, the identity (5.2) shows that the map $H - (\lambda + i\epsilon) : \mathrm{Domain}(H) \to L^2$, $\epsilon \neq 0$, is injective. Thus, the spectrum of the operator H is a subset of $\mathbb{R}$, and we have the resolvent identity

$$R_L(\lambda + i\epsilon) = (\mathrm{Id}_{W^{1,2}} + R_0(\lambda + i\epsilon)L)^{-1} R_0(\lambda + i\epsilon), \tag{5.5}$$

for $\lambda \in \mathbb{R}$ and $\epsilon \in \mathbb{R} \setminus \{0\}$.

Proof of Lemma 5.1. We show first that the operator $(\mathrm{Id}_{W^{1,2}} + R_0(\lambda + i\epsilon)L)$ is well-defined and invertible on $W^{1,2}$. Using (1.6), Lemma 4.2, and the fact that $\epsilon \neq 0$, the operator $R_0(\lambda + i\epsilon)L : W^{1,2} \to W^{1,2}$ is bounded and compact. By Fredholm's alternative, it suffices to prove that the kernel of this operator is trivial. Assume $f \in W^{1,2}$ has the property that

$$f + R_0(\lambda + i\epsilon)Lf = 0.$$

The same argument as in the proof of Lemma 4.6(a) shows that $f \equiv 0$, which completes the proof of invertibility.

Therefore, the map $\widetilde{R}_L(\lambda+i\epsilon) : L^2 \to W^{1,2}$ is a bounded operator. It remains to verify the identity (5.4). Assume $f \in L^2$ and let $g = R_0(\lambda + i\epsilon)f \in W^{2,2}$ and $h = (\mathrm{Id}_{W^{1,2}} + R_0(\lambda + i\epsilon)L)^{-1}g \in W^{1,2}$. Then

$$h = g - R_0(\lambda + i\epsilon)Lh.$$

Thus, in $\mathcal{S}'(\mathbb{R}^d)$ we have

$$\begin{aligned}[-\Delta + L - (\lambda+i\epsilon)]h = Lh &+ [-\Delta - (\lambda + i\epsilon)]g \\ &- [-\Delta - (\lambda + i\epsilon)]R_0(\lambda + i\epsilon)Lh = f,\end{aligned}$$

as desired. ∎

Proof of part (c): Assume $u \in \text{Domain}(H)$ and $Hu = \lambda u$, $\lambda \in \mathbb{R} \setminus \{0\}$. Since $u \in W^{1,2}$, we have $(\Delta + \lambda)u = Lu$, $u \in X^*$, and Theorem 1.2 applies. Thus

$$||\mu_{N,\gamma} u||_{X^*} \leq C_{N,\lambda} ||\mu_{N,\gamma} Lu||_X.$$

As in the proof of Lemma 4.4 we use (1.14) with $\varepsilon = (2C_{N,\lambda})^{-1}$ and the fact that $||\mu_{N,\gamma} u||_{X^*} < \infty$. By absorbing the term $(1/2)||\mu_{N,\gamma} u||_{X^*}$,

$$||\mu_{N,\gamma} u||_{X^*} \leq C_{N,L,\lambda} ||u||_{B^*}.$$

Part (iii) follows by letting $\gamma \to 0$.

Proof of part (b): For any $\lambda \in \mathcal{E}$ let

$$\mathcal{H}_\lambda = \{u \in \text{Domain}(H) : Hu = \lambda u\}.$$

By Lemma 4.5, $\widetilde{\mathcal{E}} \subseteq \mathcal{E}$ and $\mathcal{F}_\lambda^+ \cup \mathcal{F}_\lambda^- \subseteq \mathcal{H}_\lambda$. It suffices to show that $\mathcal{E} \subseteq \widetilde{\mathcal{E}}$ and $\mathcal{H}_\lambda \subseteq \mathcal{F}_\lambda^+ \cap \mathcal{F}_\lambda^-$. Since $\text{Domain}(H) \subseteq X^*$, it suffices to show that if $u \in \mathcal{H}_\lambda$ then

$$u + R_0(\lambda \pm i0)Lu = 0. \tag{5.6}$$

Since $(-\Delta - \lambda)u + Lu = 0$ we have

$$R_0(\lambda \pm i0)[(-\Delta - \lambda)u] + R_0(\lambda \pm i0)Lu = 0.$$

For (5.6), it suffices to prove that

$$R_0(\lambda \pm i0)[(-\Delta - \lambda)u] = u.$$

This is clear if $\lambda < 0$, for any $u \in \mathcal{S}'(\mathbb{R}^d)$. Assume $\lambda > 0$ and $R_0(\lambda \pm i0)[(-\Delta - \lambda)u] = u' \in X^*$. By Lemma 4.1(c), since $(-\Delta - \lambda)u = -Lu \subseteq X$, $(-\Delta - \lambda)u' = (-\Delta - \lambda)u$, thus

$$(-\Delta - \lambda)(u - u') = 0. \tag{5.7}$$

Since $(-\Delta - \lambda)u \in X$, and by definition of u', Lemma 4.3 gives

$$\lim_{R \to \infty} \int_{\mathbb{R}^d} |u'|^2(x)\, \phi(x/R)\, dx/R = c_2(\lambda) \int_{\sqrt{\lambda} S^{d-1}} |\widehat{(-\Delta - \lambda)u}|^2\, \sigma(d\xi). \tag{5.8}$$

Since u is rapidly decreasing in L^2 (using part (c)), it follows that $u \in L^1(\mathbb{R}^d)$, and thus $\hat{u} \in C(\mathbb{R}^d)$. Hence,

$$[(-\Delta - \lambda)u]^\wedge(\xi) = (\xi^2 - \lambda)\hat{u}(\xi)$$

both in the sense of distributions and as continuous functions. But the right-hand side vanishes on $\sqrt{\lambda} S^{d-1}$, and so the limit in (5.8) vanishes. Thus

$$\lim_{R \to \infty} R^{-1} \int_{|x| \leq R} |u'|^2\, dx = 0.$$

Using again the fact that u is rapidly decreasing in L^2, we can apply Theorem 1.2 with $N = 0$ to the function $u - u'$. The identity (5.7) gives $u \equiv u'$, which completes the proof of (5.6).

Proof of part (d): We use the resolvent identity

$$R_L(\lambda+i\epsilon) = (\mathrm{Id}_{W^{1,2}} + R_0(\lambda+i\epsilon)L)^{-1}R_0(\lambda+i\epsilon)$$

(see (5.5)). Recall that $\mathcal{E} = \widetilde{\mathcal{E}}$. The main bound (1.18) then follows from Theorem 1.1 and Lemma 4.6(b).

Proof of part (e): The statement $\sigma_{\mathrm{sc}}(H) = \emptyset$ is an immediate consequence of part (d), see [15, Theorem XIII.20]. To prove that $\sigma_{\mathrm{ac}}(H) \subseteq [0,\infty)$, assume $\lambda \in (-\infty,0)\setminus\mathcal{E}$. We have to prove that λ is in the resolvent set of H. We use Lemma 5.1. Since $\lambda \notin \sigma_{\mathrm{pp}}(H)$, the equation $(\mathrm{Id}_{W^{1,2}} + R_0(\lambda)L)f = 0$ has no solutions in $W^{1,2}$. Since the operator $R_0(\lambda)L$ is compact on $W^{1,2}$ (see the proof of Lemma 5.1), Fredholm's alternative shows that the operator $(\mathrm{Id}_{W^{1,2}} + R_0(\lambda)L)$ is invertible on $W^{1,2}$. It follows, as in Lemma 5.1, that λ is in the resolvent set of H.

The reverse inclusion $\sigma_{\mathrm{ac}}(H) \supseteq [0,\infty)$ follows from the existence of the wave operators which we establish in the next paragraph.

Proof of part (f): This will be done by means of a local version of Kato's smoothing theory. This is the only place in the proof where condition (3) of our definition of admissible perturbations is required. According to [15, Theorem XIII.31] and its corollary[3], we need to prove the following: Write $H_0 = -\Delta$, $H - H_0 = L = \sum_{j=1}^J A_j^* B_j$ as in condition (3). Then we need to show that each B_j is H_0-bounded and that each A_j is H-bounded. Furthermore, we need to show that for any compact interval I so that

$$I \subset \mathbb{R}\setminus(\mathcal{E}\cup\{0\}) \tag{5.9}$$

we have the property that $A_jE(I)$ is H-smooth and $B_jE_0(I)$ is H_0-smooth in the sense of Kato, see [9]. Here E_0 and E denote the spectral projections associated with H_0 and H, respectively.

We start with the boundedness properties, and then discuss the smoothness. Thus we need to prove that there exist constants a, b so that for each $1 \le j \le J$

$$\begin{aligned} &\mathrm{Domain}(A_j) \supseteq \mathrm{Domain}(-\Delta) \\ &\qquad \|A_jf\|_{L^2} \le a\|\Delta f\|_{L^2} + b\|f\|_{L^2} \qquad \forall\, f \in \mathrm{Domain}(\Delta) \end{aligned} \tag{5.10}$$

$$\begin{aligned} &\mathrm{Domain}(B_j) \supseteq \mathrm{Domain}(H) \\ &\qquad \|B_jf\|_{L^2} \le a\|Hf\|_{L^2} + b\|f\|_{L^2} \qquad \forall\, f \in \mathrm{Domain}(H). \end{aligned} \tag{5.11}$$

[3]Strictly speaking, [15, Theorem XIII.31] and its corollary are only stated with $J = 1$. But the same proof also applies to the case $J > 1$ needed here. Indeed, the only change is to the first inequality on page 166 of [15] which needs to be replaced with

$$\le \sum_{j=0}^J \|A_j(H-z)^{-1}\|\,\|B_j(H_0-z)^{-1}e^{-iH_0t}E_I^{(0)}\phi\|\,\|\psi\|.$$

By assumption, Domain(A_j), Domain$(B_j) \supseteq W^{1,2}$, so that the required set inclusions are clear. Furthermore, condition (3) guarantees that

$$\|A_j f\|_{L^2} + \|B_j f\|_{L^2} \le C\|f\|_{X^*} \le C\|f\|_{W^{1,2}}.$$

In conjunction with (5.1), this implies (5.10) and (5.11).

Next, we discuss the smoothness of A_j and B_j. In view of (5.9) the limiting absorption principle (1.18) holds for I, and similarly for the free resolvent R_0. It is shown in [15, Theorem XIII.30] that it suffices to prove that

$$\sup_{\lambda\in I, 0<\epsilon<1} |\epsilon| \left\| A_j R_L(\lambda+i\epsilon) \right\|^2_{L^2\to L^2} < \infty, \quad \sup_{\lambda\in I, 0<\epsilon<1} |\epsilon| \left\| B_j R_0(\lambda+i\epsilon) \right\|^2_{L^2\to L^2} < \infty$$

for the required smoothness properties of A_j and B_j to hold. However, these are easy consequences of (1.18) and (1.7), respectively, since we are requiring that $A_j, B_j : X^* \to L^2$ as bounded operators. Indeed, we only need to verify that

$$\sup_{\substack{0<\epsilon<1\\ \lambda\in I}} |\epsilon| \left\|R_L(\lambda+i\epsilon)\right\|^2_{L^2\to X^*} < \infty, \quad \sup_{\substack{0<\epsilon<1\\ \lambda\in I}} |\epsilon| \left\|R_0(\lambda+i\epsilon)\right\|^2_{L^2\to X^*} < \infty. \tag{5.12}$$

To see this, fix $\epsilon \neq 0$ and apply the resolvent indentity with $f \in X$:

$$\begin{aligned} \|R_L(\lambda+i\epsilon)f\|^2_{L^2} &= \langle R_L(\lambda+i\epsilon)^* R_L(\lambda+i\epsilon)f, f\rangle \\ &= \frac{-1}{2i\epsilon}\Big\langle \big(R_L(\lambda+i\epsilon)^* - R_L(\lambda+i\epsilon)\big)f, f\Big\rangle \\ &\le \frac{1}{2|\epsilon|}[\|R_L(\lambda+i\epsilon)f\|_{X^*} + \|R_L(\lambda-i\epsilon)f\|_{X^*}]\|f\|_X \\ &\le \frac{C}{|\epsilon|}\|f\|^2_X, \end{aligned}$$

by (1.18), and similarly for $R_0(\lambda+i\epsilon)$. Now suppose $g \in L^2$ and $f \in X$. Then this estimate implies that

$$\big|\langle R_L(\lambda+i\epsilon)g, f\rangle\big| \le C|\epsilon|^{-1/2}\, \|f\|_X \|g\|_{L^2}.$$

Thus, $R_L(\lambda+i\epsilon)g$ is an element of X^* with norm

$$|\epsilon|\, \|R_L(\lambda+i\epsilon)g\|^2_{X^*} \le C\|g\|_2^2,$$

and similarly for $R_0(\lambda+i\epsilon)g$. Hence, we are done, i.e., the wave operators

$$\Omega^\pm(H, H_0) := \text{s-lim}_{t\to\mp\infty}\, e^{itH}e^{-itH_0}, \quad \Omega^\pm(H_0, H) := \text{s-lim}_{t\to\mp\infty}\, e^{itH_0}e^{-itH}E_{a.c}$$

exist and are complete, see the aforementioned corollary in [15].

We now return to the issue of showing that $\sigma(H)\cap I \neq \emptyset$ for any nonempty interval $I \subset [0,\infty)$. Indeed, fix any such compact interval which also satisfies (5.9) and let $W_\pm, \tilde W_\pm$ be the local wave operators defined as the strong limits

$$W_\pm := \text{s-lim}_{t\to\mp\infty}\, e^{iHt}e^{-itH_0}E_0(I), \quad \tilde W_\pm := \text{s-lim}_{t\to\mp\infty}\, e^{iH_0t}e^{-itH}E(I). \tag{5.13}$$

These strong limits exist because of [15, Theorem XIII.31]. Moreover, the relations

$$W_\pm^* = \tilde{W}_\pm, \quad \tilde{W}_\pm W_\pm = E_0(I), \quad W_\pm \tilde{W}_\pm = E(I)$$

hold. Since $E_0(I) \neq 0$ by choice of I, it follows that $W_\pm$ is an isometry on the range of $E_0(I)$ and $\|W_\pm\| = 1$. Thus also $\|W_\pm^*\| = 1$. Choose any $f \in L^2$ with $W_\pm^* f \neq 0$ and observe that

$$\|E(I)f\|_{L^2}^2 = \langle W_\pm W_\pm^* f, f\rangle = \|W_\pm^* f\|_{L^2}^2 \neq 0.$$

Hence $E(I) \neq 0$, which shows that $\sigma(H) \cap I \neq \emptyset$, as claimed.

Proof of Corollary 1.5: Note that any $F \in L^{d+1}(\mathbb{R}^d)$ satisfies, by Sobolev imbedding,

$$\|FS_{1/(d+1)}f\|_{L^2} \leq C\|F\|_{L^{d+1}}\|f\|_{W^{1,2}}.$$

Therefore, $A := FS_{1/(d+1)}$ is bounded relative to both H and H_0. Moreover, since $1/2 = 1/p_d' + 1/(d+1)$,

$$\|Fg\|_{L^2} \leq \|F\|_{L^{d+1}}\|g\|_{L^{p_d'}}$$

so that by definition of X^*,

$$\|Af\|_2 \leq \|F\|_{L^{d+1}}\|f\|_{X^*}.$$

Hence, for any $I \subset \mathbb{R} \setminus (\mathcal{E} \cup \{0\})$,

$$\sup_{0<\epsilon<1} \sup_{\lambda \in I} \|FS_{1/(d+1)}R_L(\lambda + i\epsilon)S_{1/(d+1)}\overline{F}\|_{L^2 \to L^2} \leq C(I, L)\,\|F\|_{L^{d+1}}^2,$$

see (1.18) and similarly with R_0. By Kato's theorem [9], more precisely the local version of this theorem as given by [15, Theorem XIII.30], these properties imply that $FS_{1/(d+1)}$ is smoothing relative to both H and H_0, and the constants involved only depend on $\|F\|_{d+1}$. Using Kato's theory [9],

$$\int_{-\infty}^{\infty} ||FS_{1/(d+1)}[e^{itH}E(I)f||_{L^2}^2\,dt \leq C(I, L)||f||_{L^2}^2||F||_{L^{d+1}}^2,$$

which is equivalent to the bound on the first term in (1.26). For the second term we define

$$A := R^{-1/2}\mathbf{1}_{[|x| \leq R]}S_1,$$

for any $R \geq 1$ and argue as before.

6. Examples of admissible perturbations

In this section we prove Proposition 1.4. We notice first that

$$||Vf||_B \leq ||V||_Y||f||_{B^*}$$

for any $V \in Y$ and $f \in B^*$. The constants C in this section may depend on the exponent q_0 if $d = 2$. Part (c) of Proposition 1.4 is clear, directly from the definition of admissible perturbations.

For part (a) we prove the following lemma:

Lemma 6.1. *We have*

$$|| \, |V|^{1/2} S_{-1/(d+1)} f||_{L^2} \le C||M_{q_0}(V)||^{1/2}_{L^{(d+1)/2}} ||f||_{L^{p'_d}}, \; f \in L^{p'_d}, \tag{6.1}$$

$$|| \, |V|^{1/2} S_{-1} f||_{L^2} \le C||M_{q_0}(V)||^{1/2}_{Y} ||f||_{B^*}, \; f \in B^*, \tag{6.2}$$

and

$$|| \, |V|^{1/2} S_{-1} f||_{L^2} \le C|| \, |V| * K_{d,1/2}||^{1/2}_{Y} ||f||_{B^*}, \; f \in B^*. \tag{6.3}$$

Proof of Lemma 6.1. We use the fact that for $\alpha \in \{1/(d+1), 1\}$

$$|S_{-\alpha} f(x)| \le C|f| * W_\alpha(x),$$

where

$$W_\alpha(x) = |y|^{-(d-\alpha)} \mathbf{1}_{\{|y|\le 1\}} + |y|^{-(d+1)} \mathbf{1}_{\{|y|\ge 1\}}.$$

For any $s \in \mathbb{Z}^d$ let Q_s denote the cube $\{x : \sup_{i=1,\dots,d} |x_i - s_i| \le 1/2\}$. For (6.1), using the Cauchy–Schwartz inequality and fractional integration

$$\begin{aligned}
|| \, |V|^{1/2} S_{-1/(d+1)} f||^2_{L^2} &\le C \sum_{s\in\mathbb{Z}^d} \int_{Q_s} |V(x)| [|f| * W_{1/(d+1)}(x)]^2 \, dx \\
&\le C \sum_{s\in\mathbb{Z}^d} ||V||_{L^{q_0}(Q_s)} \cdot || \, |f| * W_{1/(d+1)}||^2_{L^{2q'_0}(Q_s)} \\
&\le C \sum_{s\in\mathbb{Z}^d} ||V||_{L^{q_0}(Q_s)} \Big[\sum_{s'\in\mathbb{Z}^d} || \, (\mathbf{1}_{Q_{s'}} |f|) * W_{1/(d+1)}||_{L^{2q'_0}(Q_s)} \Big]^2 \\
&\le C \sum_{s\in\mathbb{Z}^d} ||V||_{L^{q_0}(Q_s)} [\sum_{s'\in\mathbb{Z}^d} ||f||_{L^{p'_d}(Q_{s'})} (1 + |s - s'|)^{-d-1}]^2 \\
&\le C \Big[\sum_{s\in\mathbb{Z}^d} ||V||^{(d+1)/2}_{L^{q_0}(Q_s)} \Big]^{2/(d+1)} ||f||^2_{L^{p'_d}},
\end{aligned}$$

which gives (6.1). The proof of (6.2) is similar:

$$\begin{aligned}
|| \, |V|^{1/2} S_{-1} f||^2_{L^2} &\le C \sum_{s\in\mathbb{Z}^d} \int_{Q_s} |V(x)| [|f| * W_1(x)]^2 \, dx \\
&\le C \sum_{s\in\mathbb{Z}^d} ||V||_{L^{q_0}(Q_s)} \Big[\sum_{s'\in\mathbb{Z}^d} || \, (\mathbf{1}_{Q_{s'}} |f|) * W_1||_{L^{2q'_0}(Q_s)} \Big]^2 \\
&\le C \sum_{s\in\mathbb{Z}^d} ||V||_{L^{q_0}(Q_s)} [\sum_{s'\in\mathbb{Z}^d} ||f||_{L^2(Q_{s'})} (1 + |s - s'|)^{-d-1}]^2 \\
&\le C \sum_{j=0}^{\infty} (2^j \sup_{s\in\mathbb{Z}^d \cap D_j} ||V||_{L^{q_0}(Q_s)}) \cdot T_j
\end{aligned}$$

where, assuming j fixed,

$$\begin{aligned} T_j &= 2^{-j} \sum_{s\in\mathbb{Z}^d\cap D_j} [\sum_{s'\in\mathbb{Z}^d} ||f||_{L^2(Q_{s'})}(1+|s-s'|)^{-d-1}]^2 \\ &\leq C2^{-j} \sum_{s\in\mathbb{Z}^d\cap D_j} \sum_{j'=0}^{\infty} 2^{|j-j'|/10} [\sum_{s'\in\mathbb{Z}^d\cap D_{j'}} ||f||_{L^2(Q_{s'})}(1+|s-s'|)^{-d-1}]^2 \\ &\leq C2^{-j} \sum_{j'=0}^{\infty} 2^{|j-j'|/10} 2^{-2|j-j'|} ||f||^2_{L^2(D_{j'})} \leq C||f||^2_{B^*}. \end{aligned} \tag{6.4}$$

This completes the proof of (6.2). To prove (6.3), for any $s\in\mathbb{Z}^d$ let $\widetilde{Q}_s$ denote the cube $\{x : \sup_{i=1,\dots,d}|x_i - s_i| \leq 3/2\}$. We replace fractional integration with the following local bound:

$$|| \, |V|^{1/2}[|f|*(|y|^{-(d-1)}\mathbf{1}_{\{|y|\leq 1\}})]||_{L^2(Q_s)} \leq C|| \, |V|*K_{d,1/2}||^{1/2}_{L^\infty(\widetilde{Q}_s)} ||f||_{L^2(\widetilde{Q}_s)}. \tag{6.5}$$

This follows from [11, Theorem 2.3]. Using the fact that $||V||_{L^1(Q_s)} \leq C|| \, |V| * K_{d,1/2}||_{L^\infty(\widetilde{Q}_s)}$, we have

$$\begin{aligned} || \, |V|^{1/2} S_{-1} f||^2_{L^2} &\leq C \sum_{s\in\mathbb{Z}^d} \int_{Q_s} |V(x)|[|f| * W_1(x)]^2 \, dx \\ &\leq C \sum_{s\in\mathbb{Z}^d} || \, |V| * K_{d,1/2}||_{L^\infty(\widetilde{Q}_s)} [\sum_{s'\in\mathbb{Z}^d} ||f||_{L^2(Q_{s'})}(1+|s-s'|)^{-d-1}]^2 \\ &\leq C \sum_{j=0}^{\infty} (2^j \sup_{s\in\mathbb{Z}^d\cap D_j} || \, |V| * K_{d,1/2}||_{L^\infty(\widetilde{Q}_s)}) \cdot T_j, \end{aligned}$$

where T_j is as above. The bound (6.4) completes the proof of the lemma. □

We return to the proof of Proposition 1.4. Let $\mathcal{N}_1(V) = ||M_{q_0}(V)||_{L^{(d+1)/2}}$, $\mathcal{N}_2(V) = ||M_{q_0}(V)||_Y$, and $\mathcal{N}_3(V) = || \, |V| * K_{d,1/2}||_Y$. It follows from Lemma 6.1 that

$$|| \, |V|^{1/2} u||_{L^2} \leq C \min_{i\in\{1,2,3\}} \mathcal{N}_i(V)^{1/2} ||u||_{X^*} \tag{6.6}$$

for any $u \in X^*$. For potentials V as in (1.19), (1.20), or (1.21) and $u\in X^*$ we define the distribution $L_V u$ by the formula

$$\langle L_V u, \phi\rangle := \langle |V|^{1/2} u, |V|^{1/2}\mathrm{sign}(V)\phi\rangle = \int_{\mathbb{R}^d} V u \overline{\phi} \, dx. \tag{6.7}$$

The distribution $L_V u$ is well defined, in view of (6.6). Using (1.12) and (6.6)

$$||L_V u||_X \leq C \min_{i\in\{1,2,3\}} \mathcal{N}_i(V) ||u||_{X^*}, \; u\in X^*, \tag{6.8}$$

thus $L_V \in \mathcal{L}(X^*, X)$. The identity (6.7) also shows that L is symmetric in the sense on (1.13).

Next, we verify (1.14). Let $\varphi : \mathbb{R}^d \to [0, C]$ denote a smooth function supported in the ball $\{x : |x| \le 1\}$ with $\int_{\mathbb{R}^d} \varphi \, dx = 1$, and $\chi : \mathbb{R}^d \to [0, 1]$ a smooth function supported in the ball $\{x : |x| \le 2\}$ and equal to 1 in the ball $\{x : |x| \le 1\}$. For $\varepsilon \in (0, 1]$ and $r \in [1, \infty)$, let $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$ and $\chi_r(x) = \chi(x/r)$. For integers $n \ge 1$ let

$$V_n(x) = \chi_n(x)(V * \varphi_{1/n})(x).$$

We will show that

$$\begin{cases} \text{if } \mathcal{N}_1(V) < \infty & \text{then } \lim_{n\to\infty} \mathcal{N}_1(V - V_n) = 0; \\ \text{if } \mathcal{N}_2(V) < \infty & \text{then } \lim_{n\to\infty} \mathcal{N}_2(V - V_n) = 0; \\ \text{if } \lim_{\delta\to 0} || \, |V| * K_{d,\delta}||_Y = 0 & \text{then } \lim_{n\to\infty} \mathcal{N}_3(V - V_n) = 0. \end{cases} \tag{6.9}$$

Assuming (6.9), the proof of (1.14) is easy. For $i \in \{1, 2, 3\}$, given ε as in (1.14), we fix $n = n(\varepsilon)$ with the property that $\mathcal{N}_i(V - V_n) \le (\varepsilon/C)^{1/2}$, where C is the constant in (6.8). Using (6.8)

$$\begin{aligned} ||\mu_{N,\gamma} L_V u||_X &\le ||L_{V-V_n}(\mu_{N,\gamma} u)||_X + ||\mu_{N,\gamma} V_n u||_X \\ &\le \varepsilon ||\mu_{N,\gamma} u||_{X^*} + C||\mu_{N,\gamma} V_n u||_B \\ &\le \varepsilon ||\mu_{N,\gamma} u||_{X^*} + C_{V,N,\varepsilon} ||u \mathbf{1}_{\{|x|\le 2n\}}||_{L^2}, \end{aligned} \tag{6.10}$$

as desired. It remains to verify (6.9). The first two limits in (6.9) are straightforward. For the last limit fix $\varepsilon > 0$. By the definition of the space Y, there is $n_{\varepsilon,V}$ with the property that

$$||[|V| * K_{d,1/2}] \mathbf{1}_{\{|x|\ge n_{\varepsilon,V}\}}||_Y \le \varepsilon/C.$$

Then

$$||[|V - V_n| * K_{d,1/2}] \mathbf{1}_{\{|x|\ge n_{\varepsilon,V}\}}||_Y \le \varepsilon/3 \tag{6.11}$$

for any integer $n \ge 1$. The condition (1.21) shows that there is $\delta_{\varepsilon,V}$ with the property that

$$||[|V - V_n| * K_{d,\delta_{\varepsilon,V}}] \mathbf{1}_{\{|x|\le n_{\varepsilon,V}\}}||_Y \le C||V * K_{d,\delta_{\varepsilon,V}}||_Y \le \varepsilon/3 \tag{6.12}$$

for any integer $n \ge 1$. Finally, notice that the kernel $K_{d,1/2} - K_{d,\delta_{\varepsilon,V}}$ is bounded. Since $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $\lim_{n\to\infty}[V_n - V]\mathbf{1}_{\{|x|\le n_{\varepsilon,V}+1\}} = 0$ in L^1, so (6.9) follows.

To verify condition (3) let $J = 1$ and $A_1 u := |V|^{1/2} u$, $B_1 u := |V|^{1/2} \mathrm{sign}(V) u$ with domains

$$\mathrm{Domain}(A_1) = \mathrm{Domain}(B_1) := \{f \in L^2 : |V|^{1/2} f \in L^2\}.$$

It follows from (6.6) that $A_1, B_1 \in \mathcal{L}(X^*, L^2)$ and $W^{1,2} \subseteq \mathrm{Domain}(A_1) = \mathrm{Domain}(B_1)$. Also, it follows from Fatou's lemma that A_1, B_1 are closed on this domain. It remains to verify the identity (1.15). The identity is clear for $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$, in view of (6.7). For $\phi, \psi \in X^*$, we define the sequences ϕ_n and ψ_n as in the proof of Lemma 4.4. In view of (4.14), which was proved

using only conditions (1) and (2) in the definition of admissible perturbations, it suffices to prove that

$$\lim_{n\to\infty}\langle A_1\phi_n, B_1\psi_n\rangle = \langle A_1\phi, B_1\psi\rangle. \tag{6.13}$$

We may assume that $||\phi||_{X^*} = ||\psi||_{X^*} = 1$. Given $\varepsilon_0 > 0$, we fix n_0 with the property that $\min_{i\in\{1,2,3\}}\mathcal{N}_i(V - V_{n_0}) \le \varepsilon$ (using (6.9)). Using (6.6)

$$|\langle A_1\phi, B_1\psi\rangle - \int_{\mathbb{R}^d} V_{n_0}\phi\overline{\psi}\,dx| \le C\varepsilon;$$
$$|\langle A_1\phi_n, B_1\psi_n\rangle - \int_{\mathbb{R}^d} V_{n_0}\phi_n\overline{\psi}_n\,dx| \le C\varepsilon.$$

The limit (6.13) follows since $\lim_{n\to\infty}\phi_n\mathbf{1}_{\{|x|\le 2n_0\}} = \phi\mathbf{1}_{\{|x|\le 2n_0\}}$ in L^2 and $\lim_{n\to\infty}\psi_n\mathbf{1}_{\{|x|\le 2n_0\}} = \psi\mathbf{1}_{\{|x|\le 2n_0\}}$ in L^2.

We now prove part (b) of the proposition. Using part (c), we may assume that $\vec{a} = (0,\ldots,0,a)$, so

$$\vec{a}\cdot\nabla - \nabla\cdot\overline{\vec{a}} = a\partial_{x_d} - \partial_{x_d}\overline{a}.$$

We are looking to define the distribution L_a by the formula

$$\langle L_a u, \phi\rangle := \langle \omega\partial_{x_d}u, \omega^{-1}\overline{a}\phi\rangle + \langle\omega^{-1}\overline{a}u, \omega\partial_{x_d}\phi\rangle, \tag{6.14}$$

for any $u \in X^*$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$. Here

$$\omega = \sum_{j=0}^{\infty} 2^{-j/2}\omega_j\mathbf{1}_{D_j}, \tag{6.15}$$

where $\omega_j > 0$ are real numbers that will be fixed depending on the function a. The distribution L_a in (6.14) is well defined if

$$||\omega\partial_{x_d}||_{X^*\to L^2} < \infty$$

and

$$||\omega^{-1}a||_{X^*\to L^2} < \infty.$$

Since $X^* \subseteq S_{-1}B^*$, we have

$$||\omega\partial_{x_d}||_{X^*\to L^2} \le C\Big[\sum_{j=0}^{\infty}\omega_j^2\Big]^{1/2}. \tag{6.16}$$

Assume that the sequence ω_j is chosen in such a way that

$$C^{-1}\omega_j \le \omega_{j+1} \le C\omega_j \tag{6.17}$$

for any integer $j \geq 0$. Then, using (6.1) we have

$$\begin{aligned} ||\omega^{-1}au||_{L^2} &\leq C||M_{q_0}(\omega^{-2}a^2)||^{1/2}_{L^{(d+1)/2}}||u||_{X^*} \\ &\leq C||u||_{X^*}\Big[\sum_{j=0}^{\infty}\int_{D_j}[M_{q_0}(\omega^{-2}a^2)]^{(d+1)/2}\,dx\Big]^{1/(d+1)} \\ &\leq C||u||_{X^*}\Big[\sum_{j=0}^{\infty}\omega_j^{-(d+1)}[2^{j/2}||M_{2q_0}(a)||_{L^{d+1}(D_j)}]^{d+1}\Big]^{1/(d+1)}. \end{aligned} \tag{6.18}$$

Using (6.2) we have

$$\begin{aligned} ||\omega^{-1}au||_{L^2} &\leq C||M_{q_0}(\omega^{-2}a^2)||^{1/2}_{Y}||u||_{X^*} \\ &\leq C||u||_{X^*}\Big[\sum_{j=0}^{\infty}2^j||M_{q_0}(\omega^{-2}a^2)||_{L^\infty(D_j)}\Big]^{1/2} \\ &\leq C||u||_{X^*}\Big[\sum_{j=0}^{\infty}\omega_j^{-2}[2^j||M_{2q_0}(a)||_{L^\infty(D_j)}]^2\Big]^{1/2}. \end{aligned} \tag{6.19}$$

Using (6.3) we have

$$\begin{aligned} ||\omega^{-1}au||_{L^2} &\leq C||\omega^{-2}|a|^2 * K_{d,1/2}||^{1/2}_{Y}||u||_{X^*} \\ &\leq C||u||_{X^*}\Big[\sum_{j=0}^{\infty}\omega_j^{-2}[2^j||(|a|^2 * K_{d,1/2})^{1/2}||_{L^\infty(D_j)}]^2\Big]^{1/2}. \end{aligned} \tag{6.20}$$

To deal with potentials a as in (1.22), we would like to fix

$$\omega_j = C_a[2^{j/2}||M_{2q_0}(a)||_{L^{d+1}(D_j)}]^{(d+1)/(d+3)},$$

in order to optimize (6.16) and (6.18). This is not possible because of the restriction (6.17). To avoid this problem, let

$$\theta_j = \sum_{j'=0}^{\infty}2^{j'/2}||M_{2q_0}(a)||_{L^{d+1}(D_{j'})}2^{-|j-j'|} \text{ and } \omega_j = [\theta_j]^{\frac{d+1}{d+3}}\Big/\Big(\sum_{j=0}^{\infty}[\theta_j]^{p_d}\Big)^{\frac{d-1}{4(d+1)}}.$$

Clearly, $2^{j/2}||M_{2q_0}(a)||_{L^{d+1}(D_j)} \leq \theta_j$ and (6.17) holds. By (6.16) and (6.18)

$$||\omega\partial_{x_d}u||_{L^2} + ||\omega^{-1}au||_{L^2} \leq C||u||_{X^*}\Big[\sum_{j=0}^{\infty}[2^{j/2}||M_{2q_0}(a)||_{L^{d+1}(D_j)}]^{p_d}\Big]^{1/(2p_d)}. \tag{6.21}$$

Similarly, to deal with potentials as in (1.23), we let

$$\theta_j = \sum_{j'=0}^{\infty}2^{j'}||M_{2q_0}(a)||_{L^\infty(D_{j'})}2^{-|j-j'|} \text{ and } \omega_j = [\theta_j]^{1/2},$$

and it follows from (6.16) and (6.19) that

$$||\omega\partial_{x_d}u||_{L^2}+||\omega^{-1}au||_{L^2}\leq C||u||_{X^*}\Big[\sum_{j=0}^{\infty}2^j||M_{2q_0}(a)||_{L^\infty(D_j)}\Big]^{1/2}. \tag{6.22}$$

Finally, to deal with potentials as in (1.21), we let

$$\theta_j=\sum_{j'=0}^{\infty}2^{j'}||(|a|^2*K_{d,1/2})^{1/2}||_{L^\infty(D_{j'})}2^{-|j-j'|}\text{ and }\omega_j=[\theta_j]^{1/2},$$

and it follows from (6.16) and (6.20) that

$$||\omega\partial_{x_d}u||_{L^2}+||\omega^{-1}au||_{L^2}\leq C||u||_{X^*}\Big[\sum_{j=0}^{\infty}2^j||(|a|^2*K_{d,1/2})^{1/2}||_{L^\infty(D_j)}\Big]^{1/2}. \tag{6.23}$$

It follows from (6.21), (6.22) and (6.23) that the distribution L_a in (6.14) is well defined. In fact,

$$\langle L_a u,\phi\rangle=\int_{\mathbb{R}^d}a(\partial_{x_d}u)\overline{\phi}+\overline{a}u(\partial_{x_d}\overline{\phi})\,dx, \tag{6.24}$$

for any $u\in X^*$ and $\phi\in\mathcal{S}(\mathbb{R}^d)$, where the integral converges absolutely. Let $\mathcal{N}_1'(a)=\Big[\sum_{j=0}^{\infty}(2^{j/2}||M_{2q_0}(a)||_{L^{d+1}(D_j)})^{p_d}\Big]^{1/p_d}$, $\mathcal{N}_2'(a)=||M_{2q_0}(a)||_Y$, and $\mathcal{N}_3'=||(|a|^2*K_{d,1/2})^{1/2}||_Y$. Using (1.12), it follows from (6.14), (6.21), (6.22), and (6.23) that

$$||L_a u||_X\leq C||u||_{X^*}\min_{i\in\{1,2,3\}}\mathcal{N}_i'(a), \tag{6.25}$$

Thus $L_a\in\mathcal{L}(X^*,X)$. It follows easily from (6.14) that L_a is symmetric, in the sense of (1.13).

To prove condition (2) in the definition of admissible perturbations, let

$$a_n(x)=\chi_n(x)(a*\varphi_{1/n})(x),$$

where χ and φ are as before. As in the proof of (6.9), it follows that

$$\begin{cases}\text{if }\mathcal{N}_1'(a)<\infty & \text{then }\lim_{n\to\infty}\mathcal{N}_1'(a-a_n)=0;\\ \text{if }\mathcal{N}_2'(a)<\infty & \text{then }\lim_{n\to\infty}\mathcal{N}_2'(a-a_n)=0;\\ \text{if }\lim_{\delta\to 0}||\,[|a|^2*K_{d,\delta}]^{1/2}||_Y=0 & \text{then }\lim_{n\to\infty}\mathcal{N}_3'(a-a_n)=0.\end{cases}$$

The identity (6.24) shows that $L_a=L_{a_n}+L_{a-a_n}$. An estimate similar to (6.10) shows that it suffices to prove that for any $N\geq 0$

$$||\mu_{N,\gamma}L_a u||_X\leq C_N\min_{i\in\{1,2,3\}}\mathcal{N}_i'(a)||\mu_{N,\gamma}u||_{X^*}, \tag{6.26}$$

for any $u\in X^*$ and any $\gamma\in(0,1]$. With the notation in Lemma 3.1, it is easy to check that

$$\mu_{N,\gamma}L_a u=L_a(\mu_{N,\gamma}u)-b_d(a-\overline{a})(\mu_{N,\gamma}u).$$

The bound for the first term follows directly from (6.25). For the second term, we define $\omega = \omega(a) = \omega(\overline{a})$ as before, and use (1.12), (6.21), (6.22), and (6.23):

$$\begin{aligned}||b_d(a-\overline{a})(\mu_{N,\gamma}u)||_X &\leq C \sup_{\phi\in\mathcal{S}(\mathbb{R}^d),\,||\phi||_{X^*}=1} |\langle \omega^{-1}(a-\overline{a})(\mu_{N,\gamma}u), \omega b_d\phi\rangle| \\ &\leq C||\mu_{N,\gamma}u||_{X^*} \min_{i\in\{1,2,3\}} \mathcal{N}_i'(a)^{1/2} \sup_{\phi\in\mathcal{S}(\mathbb{R}^d),\,||\phi||_{X^*}=1} ||\omega b_d\phi||_{L^2} \\ &\leq C||\mu_{N,\gamma}u||_{X^*} \min_{i\in\{1,2,3\}} \mathcal{N}_i'(a),\end{aligned}$$

using (3.1). This completes the proof of (1.14).

For condition (3) in the definition of admissible perturbations, let $J = 2$ and

$$A_1u = B_2u := \omega^{-1}\overline{a}u, \text{ and } B_1u = A_2u := \omega\partial_{x_d}u,$$

where ω is defined as before. As domains we again choose the natural ones, i.e.,

$$\begin{aligned}\text{Domain}(A_1) = \text{Domain}(B_2) &:= \{f \in L^2 : \omega^{-1}\overline{a}f \in L^2\} \\ \text{Domain}(B_1) = \text{Domain}(A_2) &:= \{f \in L^2 : \omega\partial_{x_d}f \in L^2\}.\end{aligned}$$

It is again easy to see that these domains make A_1, A_2, B_1, B_2 closed.[4] It follows from (6.21), (6.22), and (6.23) that A_1, B_1, A_2, $B_2 \in \mathcal{L}(X^*, L^2)$. To verify the identity (1.15), we notice first that the identity holds if $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$, in view of the definition (6.14). The proof for $\phi, \psi \in X^*$ then follows by the same limiting argument as in part (a), see the proof of (6.13).

References

[1] S. Agmon, *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **2** (1975), 151–218.

[2] H. Cycon, R. Froese, W. Kirsch and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry.* Texts and Monographs in Physics. Springer Study Edition. Springer-Verlag, Berlin, 1987.

[3] S. A. Denisov and A. Kiselev, *Spectral Properties of Schrödinger Operators with Decaying Potentials.*, preprint 2005.

[4] I. M. Gelfand and G. E. Shilov, *Generalized Functions*, Vol. I, Academic Press, New York (1964).

[5] M. Goldberg and W. Schlag, *A limiting absorption principle for the three-dimensional Schrödinger equation with L^p potentials*, Int. Math. Res. Not. 2004, no. 75, 4049–4071.

[6] L. Hörmander, *The Analysis of Linear Partial Differential Operators,* Grundlehren der Mathematischen Wissenschaften, Springer–Verlag, Berlin (1985).

[7] A. D. Ionescu and D. Jerison, *On the absence of positive eigenvalues of Schrödinger operators with rough potentials*, Geom. Funct. Anal. **13** (2003), 1029–1081.

[8] T. Kato, *Growth properties of solutions of the reduced wave equation with a variable coefficient*, Comm. Pure Appl. Math. **12** (1959), 403–425.

[4] Strictly speaking, one should smooth out $\mathbf{1}_{D_j}$ in (6.15) here, but we ignore this issue.

[9] T. Kato, *Wave operators and similarity for some non-selfadjoint operators*, Math. Ann. **162** (1965/1966), 258–279.

[10] C. E. Kenig, A. Ruiz and C. D. Sogge, *Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators*, Duke Math. J. **55** (1987), 329–347.

[11] R. Kerman and E. T. Sawyer, *The trace inequality and eigenvalue estimates for Schrödinger operators*, Ann. Inst. Fourier (Grenoble) **36** (1986), 207–228.

[12] A. Kiselev, *Imbedded singular continuous spectrum for Schrödinger operators*, J. Amer. Math. Soc. 18 (2005), no. 3, 571–603.

[13] H. Koch, D. Tataru *Carleman estimates and absence of embedded eigenvalues*, preprint 2005.

[14] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness,* Academic Press, New York-London, 1975.

[15] M. Reed and B. Simon *Methods of modern mathematical physics. IV. Analysis of operators,* Academic Press, New York-London, 1978.

[16] A. Ruiz and L. Vega, *On local regularity of Schrödinger equations*, Int. Math. Res. Not. **1993** (1993), 13–27.

[17] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. **7** (1982), 447–526.

[18] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals,* Princeton Mathematical Series, 71, Princeton University Press, Princeton, NJ (1993).

[19] P. A. Tomas, *A restriction theorem for the Fourier transform*, Bull. Amer. Math. Soc. **81** (1975), 477–478.

Department of Mathematics, University of Wisconsin–Madison, Madison, WI 53706, U.S.A.

E-mail address: ionescu@math.wisc.edu

Department of Mathematics, The University of Chicago, 5734 South University Avenue, Chicago, IL 60637, U.S.A.

E-mail address: schlag@math.uchicago.edu